(1) If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathcal{D}(T)$ such that $\|x\| < 1$ we have the strict inequality $\|Tx\| < \|T\|$.

► A linear operator T is a bounded if $\exists C > 0$ s.t.

$$\|Tx\| \leq C \cdot \|x\|, \quad \forall x \in \mathcal{D}(T).$$

We know that

$$\|T\| = \min\{C : \|Tx\| \leq C\|x\|, \ \forall x \in \mathcal{D}(T)\}.$$

Then

$$\|Tx\| \leq \|T\| \cdot \|x\|.$$

Since $\|x\| < 1$, then $\|Tx\| \leq \|T\| \cdot \underbrace{\|x\|}_{<1} < \|T\|$. ◄

(2) Let $T: \mathcal{D}(T) \to Y$ be a linear operator. Show that

$$\|T\| := \sup_{\substack{x \in \mathcal{D}(T),\\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in \mathcal{D}(T),\\ \|x\|=1}} \|Tx\|.$$

▶ Let us show that:

1) $$\sup_{\substack{x \in \mathcal{D}(T)\\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \geq \sup_{\substack{x \in \mathcal{D}(T)\\ \|x\|=1}} \|Tx\| ;$$

2) $$\sup_{\substack{x \in \mathcal{D}(T)\\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \leq \sup_{\substack{x \in \mathcal{D}(T)\\ \|x\|=1}} \|Tx\| .$$

1) The set $\{x \in \mathcal{D}(T) : \|x\| = 1\} \subset \{x \in \mathcal{D}(T) : x \neq 0\}$. Then

$$\sup_{\substack{x \in \mathcal{D}(T)\\ \|x\|=1}} \|Tx\| = \sup_{\substack{x \in \mathcal{D}(T)\\ \|x\|=1}} \frac{\|Tx\|}{\|x\|} \leq \sup_{\substack{x \in \mathcal{D}(T)\\ x \neq 0}} \frac{\|Tx\|}{\|x\|} . \quad !$$

by the prop. of sup

2) From the definition of suprimum we know that $\forall \varepsilon > 0 \ \exists \, \tilde{x} \in \mathcal{D}(T)$, $\tilde{x} \neq 0$, s.t.

$$\|T\| - \varepsilon \leq \frac{\|T\tilde{x}\|}{\|\tilde{x}\|} \leq \|T\| = \sup_{\substack{x \in \mathcal{D}(T)\\ x \neq 0}} \frac{\|Tx\|}{\|x\|} .$$

Take $y = \frac{\tilde{x}}{\|\tilde{x}\|}$, then $\|y\| = 1$, $\|Ty\| = \left\|T \frac{\tilde{x}}{\|\tilde{x}\|}\right\| = \frac{\|T\tilde{x}\|}{\|\tilde{x}\|}$

and $$\|Ty\| \leq \sup_{\substack{x \in \mathcal{D}(T)\\ \|x\|=1}} \|Tx\| .$$

Hence,

$$\|T\| - \varepsilon \leq \frac{\|T\tilde{x}\|}{\|\tilde{x}\|} = \|Ty\| \leq \sup_{\substack{x \in \mathcal{D}(T)\\ \|x\|=1}} \|Tx\|$$

$$\Rightarrow \|T\| - \varepsilon \leq \sup_{\substack{x \in \mathcal{D}(T)\\ \|x\|=1}} \|Tx\| \Rightarrow \|T\| \leq \sup_{\substack{x \in \mathcal{D}(T)\\ \|x\|=1}} \|Tx\| , \ \varepsilon \to 0.$$

From 1) and 2) we have that

$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T)\\ \|x\|=1}} \|Tx\| .$$

◀

(3) Show that the operator $T: \ell^\infty \to \ell^\infty$ defined by $Tx = (\eta_k)_{k\geq 1}$, $\eta_k = \frac{\xi_k}{k}$, $k \geq 1$, $x = (\xi_k)_{k\geq 1}$, is linear and bounded. Compute its norm.

▸ • Check that T is linear:

a) $\forall x, y \in \ell^\infty \quad T(x+y) = Tx + Ty$

b) $\forall x \in \ell^\infty, \forall \alpha \in K \quad T(\alpha x) = \alpha Tx.$

Let $x = (\xi_k)_{k\geq 1}$, $y = (\tau_k)_{k\geq 1}$

a) $T(x+y) = \left(\frac{\xi_1+\tau_1}{1}, \frac{\xi_2+\tau_2}{2}, \dots, \frac{\xi_n+\tau_n}{n}, \dots\right) =$

$= \left(\frac{\xi_1}{1}, \frac{\xi_2}{2}, \dots, \frac{\xi_n}{n}, \dots\right) + \left(\frac{\tau_1}{1}, \frac{\tau_2}{2}, \dots, \frac{\tau_n}{n}, \dots\right) =$

$= Tx + Ty.$

b) $T(\alpha x) = \left(\frac{\alpha\xi_1}{1}, \frac{\alpha\xi_2}{2}, \dots, \frac{\alpha\xi_n}{n}, \dots\right) = \alpha\left(\frac{\xi_1}{1}, \frac{\xi_2}{2}, \dots, \frac{\xi_n}{n}, \dots\right) =$

$= \alpha Tx$

$\Rightarrow T$ is linear.

• Show that T is bounded. Recall that T is bounded if $\exists c > 0$ s.t.

$$\|Tx\| \leq c \cdot \|x\|.$$

Take $\forall x \in \ell^\infty$. Then

$$\|Tx\|_{\ell^\infty} = \sup_{k\geq 1} \left|\frac{\xi_k}{k}\right| \leq \sup_{k\geq 1} |\xi_k| = \|x\|_{\ell^\infty} \Rightarrow$$

$\Rightarrow \exists\, C = 1$ s.t. $\|Tx\| \leq 1 \cdot \|x\|, \ \forall x \in \ell^\infty$

• Compute its norm.

Since $\|Tx\| \leq 1 \cdot \|x\|$, then $\|T\| \leq 1$. $(*)$

Show that $\|T\| = 1$.

Take $x = (1, 1, \dots, 1, \dots)$. Then $\|x\| = \sup_{k\geq 1} |\xi_k| = \sup_{k\geq 1} 1 = 1$,

$\|Tx\| = \sup_{k\geq 1} \left|\frac{\xi_k}{k}\right| = \sup_{k\geq 1} \frac{1}{k} = 1 \Rightarrow$

$\Rightarrow \frac{\|Tx\|}{\|x\|} = 1.$

Since $\|T\| \geq \frac{\|Tx\|}{\|x\|}$, then $\|T\| \geq 1$. $(**)$

From $(*)$ and $(**)$ we obtain that $\|T\| = 1$.

④ Compute the norm of the linear operator
$T: C[0;1] \to C[0;1]$

$$(Tx)(t) = \int_0^t s\, x(s) ds, \quad t \in [0;1].$$

$$\|Tx\| = \max_{t\in[0;1]} \left| \int_0^t s\, x(s) ds \right| \le \max_{t\in[0;1]} \int_0^t s \cdot |x(s)| ds \le$$

$$\le \max_{t\in[0;1]} \int_0^t s \cdot \underbrace{\max_{s\in[0;1]} |x(s)|}_{=\|x\|} ds = \|x\| \cdot \max_{t\in[0;1]} \int_0^t s\, ds = \|x\| \cdot \max_{t\in[0;1]} \frac{t^2}{2} =$$

$$= \frac{1}{2} \cdot \|x\|, \quad \forall x \in C[0;1]$$

$\Rightarrow$ T is bounded.

$$\frac{\|T(x)\|}{\|x\|} \le \frac{1}{2} \Rightarrow \|T\| = \sup_{\substack{x\in C[0;1] \\ x \ne 0}} \frac{\|Tx\|}{\|x\|} \le \frac{1}{2}. \quad (*)$$

Let us show that $\|T\| = \frac{1}{2}$.

Take $x(t) = 1$, $t \in [0;1]$. Then $\|x\| = \max_{t\in[0;1]} |x(t)| = 1$,

$$(Tx)(t) = \int_0^t s \cdot 1\, ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

and

$$\|Tx\| = \max_{t\in[0;1]} \left|\frac{t^2}{2}\right| = \frac{1}{2} \Rightarrow \|Tx\| = \frac{1}{2} \underbrace{\|x\|}_{=1} \Rightarrow \frac{\|Tx\|}{\|x\|} = \frac{1}{2}.$$

$$\Rightarrow \|T\| \ge \frac{\|Tx\|}{\|x\|} = \frac{1/2}{1} = \frac{1}{2}. \quad (*)$$

From (*) and (**) we have that

$$\frac{1}{2} \le \|T\| \le \frac{1}{2}$$

$$\Rightarrow \|T\| = \frac{1}{2}.$$

5 Using the definition, compute the norm of the following functional f on c_0

$$f(x) = \sum_{k=1}^{\infty} \frac{\xi_k}{3^k}, \quad x = (\xi_k)_{k \geq 1} \in c_0.$$

▸ Let us show that for $\forall x = (\xi_k)_{k\geq 1} \in c_0$

$$|f(x)| \leq C \cdot \|x\|.$$

$$|f(x)| = \left|\sum_{k=1}^{\infty} \frac{\xi_k}{3^k}\right| \leq \sum_{k=1}^{\infty} \frac{|\xi_k|}{3^k} \leq \sum_{k=1}^{\infty} \frac{\sup_{k\geq 1}|\xi_k|}{3^k} = \|x\| \cdot \sum_{k=1}^{\infty} \frac{1}{3^k} =$$

$$= \|x\| \cdot \frac{1/3}{1-\frac{1}{3}} = \|x\| \cdot \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2} \cdot \|x\|, \quad \forall x \in c_0.$$

$$\Rightarrow \frac{|f(x)|}{\|x\|} \leq \frac{1}{2} \Rightarrow \|f\| = \sup_{\substack{x \in l_2(T) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \leq \frac{1}{2}. \qquad (*)$$

Let us find $x_n \in c_0$ s.t.

$$|f(x_n)| = C_n \cdot \|x_n\|, \quad x_n = (\xi_k^n)_{k\geq 1}, \; n \geq 1.$$

Take
$$x_1 = (1, 0, 0, \dots)$$
$$x_2 = (1, 1, 0, \dots)$$
$$\text{- - - - - - - -}$$
$$x_n = (\underbrace{1, 1, \dots, 1}_{n}, 0, \dots)$$
$$\text{- - - - - - - - - -}$$

$\Rightarrow x_n = (\xi_k^n)_{k\geq 1} \in c_0$, because $\lim_{n\to\infty} \xi_k^n = 0$.

$$\|x_n\| = \sup_{k\geq 1} |\xi_k^n| = 1.$$

Then

$$|f(x_n)| = \left|\sum_{k=1}^{\infty} \frac{\xi_k}{3^k}\right| = \left|\sum_{k=1}^{n} \frac{\overset{=1}{\xi_k}}{3^k}\right| = \sum_{k=1}^{n} \frac{1}{3^k} = \frac{\frac{1}{3} \cdot \frac{1}{3^n} - \frac{1}{3}}{\frac{1}{3} - 1} =$$

$$= \frac{1}{3} \cdot \frac{\frac{1}{3^n} - 1}{-\frac{2}{3}} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right).$$

Set $C_n = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$.

$$|f(x_n)| = C_n \cdot \underbrace{\|x_n\|}_{=1} \Rightarrow \frac{|f(x_n)|}{\|x_n\|} = C_n \quad \text{and}$$

$$\|T\| \geq \frac{|f(x_n)|}{\|x_n\|} = C_n. \qquad (**)$$

From $(*)$ and $(**)$ we have that

$$C_n \leq \|T\| \leq \frac{1}{2}.$$

Since $C_n \to \frac{1}{2}$, $n \to \infty$ $\left(\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{1}{2}\left(1 - \underbrace{\frac{1}{3^n}}_{\to 0}\right) = \frac{1}{2}\right)$, then

$$\|T\| = \frac{1}{2}. \qquad ◀$$

⑥ Compute norms of the following functionals:

a) $f(x) = \int_0^1 t^3 x(t)\,dt$ on $L^4[0;1]$.

$(L^p[a,b])' = L^q[a;b]$, $1<p<q$, $\frac{1}{p}+\frac{1}{q}=1$.

$f \in L^p[a,b]$, $f(x) = \int_a^b u(t)x(t)\,dt$, where $u \in L^q[a,b]$.

Then $\|f\| = \|u\|_{L^q}$, $\|u\|_{L^q} = \left(\int_a^b |u(t)|^q dt\right)^{1/q}$.

We have that $p=4$, then $\frac{1}{q} = 1-\frac{1}{p} = 1-\frac{1}{4} = \frac{3}{4} \Rightarrow q = \frac{4}{3}$.

$u(t) = t^3$, $t \in [0;1]$.

$\|f\| = \|u\|_{L^{4/3}} = \left(\int_0^1 |t^3|^{4/3} dt\right)^{3/4} = \left(\int_0^1 t^4 dt\right)^{3/4} = \left(\frac{t^5}{5}\Big|_0^1\right)^{3/4} = \left(\frac{1}{5}\right)^{3/4}$.

Hence,
$\|f\| = \left(\frac{1}{5}\right)^{3/4}$.

b) $f(x) = \sum_{k=1}^{\infty} \frac{\xi_k}{\sqrt{k!}}$ on ℓ^2.

$(\ell^p)' = \ell^q$, $\frac{1}{p}+\frac{1}{q}=1$

$f \in \ell^p$, $f(x) = \sum_{k=1}^{\infty} \gamma_k \xi_k$, $x = (\xi_k)_{k=1}^{\infty}$.

Then $\|f\| = \|u\|_{\ell^q}$, where $u = (\gamma_k)_{k=1}^{\infty} \in \ell^q$, and

$\|u\|_{\ell^q} = \left(\sum_{k=1}^{\infty} |\gamma_k|^q\right)^{1/q}$.

We have that $p=2$, then $q=2$.

$f(x) = \sum_{k=1}^{\infty} \frac{\xi_k}{\sqrt{k!}} \Rightarrow u = (\gamma_k)_{k=1}^{\infty} = \left(\frac{1}{\sqrt{k!}}\right)_{k=1}^{\infty}$.

$\|f\| = \|u\|_{\ell^2} = \left(\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{k!}}\right)^2\right)^{1/2} = \left(\sum_{k=1}^{\infty}\frac{1}{k!}\right)^{1/2} =$

$= \left(\sum_{k=0}^{\infty}\frac{1}{k!} - 1\right)^{1/2}$.

Since $e^z = \sum_{k=0}^{\infty} \frac{z^n}{n!}$, then $\sum_{k=0}^{\infty}\frac{1}{k!} = \sum_{k=0}^{\infty}\frac{1^k}{k!} = e^1$

and $\|f\| = \left(\sum_{k=0}^{\infty}\frac{1}{k!} - 1\right)^{1/2} = (e^1-1)^{1/2} = \sqrt{e-1}$.

⑦ Find the norm of the functional defined on $C[-1;1]$ by
$$f(x)=\int_{-1}^{0}x(t)\,dt-2\int_{0}^{1}x(t)\,dt.$$

▸ Let us find the norm of the functional using the defin.

$$|f(x)|=\left|\int_{-1}^{0}x(t)\,dt-2\int_{0}^{1}x(t)\,dt\right|\le\int_{-1}^{0}|x(t)|\,dt+2\int_{0}^{1}|x(t)|\,dt\le$$
$$\le\int_{-1}^{0}\underbrace{\max_{t\in[-1;1]}|x(t)|}_{=\|x\|}dt+2\int_{0}^{1}\underbrace{\max_{t\in[-1;1]}|x(t)|}_{=\|x\|}dt=\|x\|\left(\int_{-1}^{0}dt+2\int_{0}^{1}dt\right)=$$
$$=\|x\|\cdot(1+2)=3\|x\|$$
$$\Rightarrow\ \frac{|f(x)|}{\|x\|}\le 3\quad\forall x\in C[-1;1]$$
$$\Rightarrow\ \|f\|=\sup_{\substack{x\in C[-1;1]\\ x\neq 0}}\frac{|f(x)|}{\|x\|}\le 3.\qquad(*)$$

Let us show that $|f(x_n)|=C_n\cdot\|x_n\|$.

Set
$$x_n(t)=\begin{cases}1, & -1\le t\le -\frac{1}{n},\\ -nt, & -\frac{1}{n}<t\le\frac{1}{n},\\ -1, & \frac{1}{n}<t\le 1,\end{cases}$$

$x_n\in C[-1,1]$, because $\underbrace{\lim_{t\to-\frac{1}{n}-0}x_n(t)}_{1}=\lim_{t\to-\frac{1}{n}+0}x_n(t)=\lim_{t\to-\frac{1}{n}+0}(-nt)=1,$
$$\lim_{t\to\frac{1}{n}-0}x_n(t)=\lim_{t\to\frac{1}{n}+0}x_n(t)=-1.$$

$$\|x_n\|=\max_{t\in[-1;1]}|x_n(t)|=1.$$
$$|f(x_n)|=\left|\int_{-1}^{-1/n}1\,dt+\int_{-\frac{1}{n}}^{0}(-nt)\,dt-2\left(\int_{0}^{1/n}(-nt)\,dt+\int_{\frac{1}{n}}^{1}(-1)\,dt\right)\right|=$$
$$=\left|\,t\,\Big|_{-1}^{-1/n}-\frac{nt^2}{2}\Big|_{-\frac{1}{n}}^{0}+\frac{2nt^2}{2}\Big|_{0}^{1/n}+2t\Big|_{\frac{1}{n}}^{1}\right|=\left|-\frac{1}{n}+1+\frac{1}{2n}+\frac{1}{n}+2-\frac{2}{n}\right|=$$
$$=\left|3+\frac{1-4}{2n}\right|=\left|3-\frac{3}{2n}\right|=3-\frac{3}{2n}.$$

Let $C_n=3-\frac{3}{2n}$, then $|f(x_n)|=C_n\cdot\underbrace{\|x_n\|}_{=1}\Rightarrow\dfrac{|f(x_n)|}{\|x_n\|}=C_n$

and $\|f\|\ge\dfrac{|f(x_n)|}{\|x_n\|}=C_n=3-\dfrac{2}{2n}.\qquad(**)$

We have by $(*)$ and $(**)$ that
$$3-\frac{3}{2n}\le\|f\|\le 3,$$
$\lim\limits_{n\to\infty}C_n=\lim\limits_{n\to\infty}\left(3-\frac{3}{2n}\right)=3.$ Hence, $\|f\|=3.$ ◀

(8) Show that a linear functional is continuous on a normed space if and only if is kernel is closed.

$\Rightarrow$) Let f is continuous. Show that $\operatorname{Ker} f$ is closed.
Since f is cont. and $\{0\}$ is closed, then $f^{-1}(\{0\})$ is closed.

$\Leftarrow$) Let $\operatorname{Ker} f$ is closed. Need to prove that f is continuous.
Assume that f is not continuous. Then f is not bounded, that is,

$$\forall C \quad \exists x \in X : \quad |f(x)| \geq C \|x\|.$$

For a sequence $x_n \in \operatorname{Ker} f$ s.t. $\|x\| = 1$

$$|f(x_n)| \geq C \|x_n\| = C.$$

Take $x \notin \operatorname{Ker} f$.
Set $\tilde{x}_n = x - \frac{f(x)}{f(x_n)} \cdot x_n$. Then

$$f(\tilde{x}_n) = f\left(x - \frac{f(x)}{f(x_n)} x_n\right) = f(x) - \frac{f(x)}{f(x_n)} \cdot f(x_n) = f(x) - f(x) = 0$$

$\Rightarrow \tilde{x}_n \in \operatorname{Ker} f$.

Show that $\tilde{x}_n \to x$. Indeed,

$$\|x_n - x\| = \left\|x - \frac{f(x)}{f(x_n)} x_n - x\right\| = \left\|\frac{f(x)}{f(x_n)} x_n\right\| =$$

$$= \underbrace{\left|\frac{f(x)}{f(x_n)}\right|}_{\to 0,\ \text{because } f(x_n) \to \infty,\ n \to \infty} \cdot \underbrace{\|x\|}_{=1} \to 0$$

We have that $\tilde{x}_n \in \operatorname{Ker} f$ and $\tilde{x}_n \to x$, $x \notin \operatorname{Ker} f$.
It implies that $\operatorname{Ker} f$ is not closed.
We got the contradiction.