① Let X, X' be sets, and $\mathcal{F}'$ be a σ-algebra on X'.
[3p] Let also $f: X \to X'$ be a function.
Show that the class of sets
$$f^{-1}(\mathcal{F}') := \{f^{-1}(A') : A' \in \mathcal{F}'\}$$
is a σ-algebra on X.

► We need to show that the class $f^{-1}(\mathcal{F}')$ is a σ-algebra on X. For this let us check the following conditions (the defin. of σ-alg.)

1) $X \in f^{-1}(\mathcal{F}')$;
2) $A_1, A_2, \ldots \in f^{-1}(\mathcal{F}') \Rightarrow \bigcup_{n=1}^{\infty} A_n \in f^{-1}(\mathcal{F}')$;
3) $A \in f^{-1}(\mathcal{F}') \Rightarrow A^c = X \setminus A \in f^{-1}(\mathcal{F}')$.

1) Show that $X \in f^{-1}(\mathcal{F}')$. Take $X' \in \mathcal{F}'$

$$f^{-1}(\underbrace{X'}_{\in \mathcal{F}'}) = \{x \in X : f(x) \in X'\} = X$$

$\Rightarrow \underline{X \in f^{-1}(\mathcal{F}')}$.

2) Let $A_1, A_2, \ldots \in f^{-1}(\mathcal{F}')$ and show that $\bigcup_{n=1}^{\infty} A_n \in f^{-1}(\mathcal{F}')$
For $\forall A_n$, $n \in \mathbb{N}$, $\exists A_n' \in \mathcal{F}'$ such that $f^{-1}(A_n') = A_n$.
Then
$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} f^{-1}(A_n') = f^{-1}(\underbrace{\bigcup_{n=1}^{\infty} A_n'}_{\in \mathcal{F}' \text{ because } \mathcal{F}' \text{ is a } \sigma\text{-alg. no } X'})$$

$\Rightarrow \underline{\bigcup_{n=1}^{\infty} A_n \in f^{-1}(\mathcal{F}')}$.

3) Show that $A^c \in f^{-1}(\mathcal{F}')$.
Let $A \in f^{-1}(\mathcal{F}')$, then $\exists A' \in \mathcal{F}'$ s.t. $f^{-1}(A') = A$
Consider
$$A^c = X \setminus A = f^{-1}(X') \setminus f^{-1}(A') = f^{-1}(X' \setminus A') = f^{-1}(\underbrace{A'^c}_{\in \mathcal{F}' \text{ because } \mathcal{F}' \text{ is a } \sigma\text{-alg. on } X'})$$

$\Rightarrow \underline{A^c \in f^{-1}(\mathcal{F}')}$.

Hence, $f^{-1}(\mathcal{F}')$ is a σ-algebra on X. ◢

(2) Prove that every Borel measurable function $f: \mathbb{R} \to \mathbb{R}$ is also Lebesgue measurable.[1]

[1] A function $f: \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if it is S-measurable function, where S is the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$.

▶

f is Borel measurable if

$$\forall A' \in \mathcal{B}(\mathbb{R}) \quad f^{-1}(A') \in \underbrace{\mathcal{B}(\mathbb{R})}_{\subseteq S} \qquad (*)$$

f is S-measurable if

$$\forall A' \in \mathcal{B}(\mathbb{R}) \quad f^{-1}(A') \in S$$

According to section 4 of Lecture 4 we know that

$$\mathcal{B}(\mathbb{R}) \subseteq S \qquad (**)$$

$$\Rightarrow f^{-1}(A') \underset{\text{by } (*)}{\in} \mathcal{B}(\mathbb{R}) \underset{\text{by } (**)}{\subset} S \Rightarrow f^{-1}(A_1') \in S \quad \forall A' \in \mathcal{B}(\mathbb{R}).$$

Hence, every Borel measurable function is also Lebesgue measurable. ◀

(3) Let $f_k : X \to \mathbb{R}$, $k = 1, \dots, m$ be $\mathcal{F}$-measurable functions.
[4p] Consider $f(x) := (f_1(x), \dots, f_m(x))$, $x \in X$. Show that the function $f: X \to \mathbb{R}^m$ is also $\mathcal{F}$-measurable, that is, $f^{-1}(A') \in \mathcal{F}$ for every $A' \in \mathcal{B}(\mathbb{R}^m)$.

Given: $f_k : X \to \mathbb{R}$ be $\mathcal{F}$-measurable functions, that is,

$$\forall A' \in \mathcal{B}(\mathbb{R}) \quad f_k^{-1}(A') \in \mathcal{F}.$$

We need to show that the function $f: X \to \mathbb{R}^m$ is $\mathcal{F}$-measurable, that is,

$$\forall A' \in \mathcal{B}(\mathbb{R}^m) \quad f^{-1}(A') \in \mathcal{F}.$$

Let $A' = (a_1; b_1] \times (a_2; b_2] \times \dots \times (a_m; b_m]$, $a_i < b_i$, $i = \overline{1,m}$.
It is enough to check the property of measurability only for rectangle, because the sets of the form $A' = (a_1; b_1] \times (a_2; b_2] \times \dots \times (a_m; b_m]$, $a_i < b_i$, $i = \overline{1,m}$, generate the Borel σ-algebra on $\mathbb{R}^m$.

$$f^{-1}(A') = \{x \in X : f(x) \in A'\} =$$

$$= \{x \in X : f_1(x) \in (a_1; b_1], f_2(x) \in (a_2; b_2], \dots, f_m(x) \in (a_m; b_m]\} =$$

$$= \bigcap_{k=1}^{m} \{x \in X : f_k(x) \in (a_k; b_k]\} =$$

$$= \bigcap_{k=1}^{m} \underbrace{f^{-1}(\underbrace{(a_k; b_k]}_{\in \mathcal{B}(\mathbb{R})})}_{\in \mathcal{F} \text{ because } f_k \text{ is } \mathcal{F}\text{-meas.}} \Rightarrow \bigcap_{k=1}^{m} f^{-1}((a_k; b_k]) \in \mathcal{F}, \text{ because } \mathcal{F} \text{ is a } \sigma\text{-alg.}$$

Hence, $\forall A' \in \mathcal{B}(\mathbb{R}^m)$ $f^{-1}(A') \in \mathcal{F}$. ◀

(4) Let $f, g : X \to \mathbb{R}$ be $\mathcal{F}$-measurable. Show that
[3 p.] $\{x \in X : f(x) \geq g(x)\}$ and $\{x \in X : f(x) = g(x)\}$ belongs to $\mathcal{F}$.

1. Show that $\{x \in X : f(x) \geq g(x)\}$ belongs to $\mathcal{F}$.
Consider the function

$$h(x) = f(x) - g(x).$$

The functions f and g are $\mathcal{F}$-measurable (given).
Then h is also $\mathcal{F}$-measurable as a difference of $\mathcal{F}$-measurable functions (by Th. 6.7).
It means that

$$\{x \in X : f(x) \geq g(x)\} = \{x \in X : f(x) - g(x) \geq 0\} =$$

$$= \{x \in X : h(x) \geq 0\} = h^{-1}([0; +\infty)).$$

h is $\mathcal{F}$-measurable, then $h^{-1}(\underbrace{[0; +\infty)}_{\in \mathcal{B}(\mathbb{R})}) \in \mathcal{F}$ by defin. of $\mathcal{F}$-meas. func.

Hence, $\{x \in X : f(x) \geq g(x)\} \in \mathcal{F}$.

2. Show that $\{x \in X : f(x) = g(x)\} \in \mathcal{F}$.
Similarly, let

$$p(x) = f(x) - g(x).$$

p is $\mathcal{F}$-measurable by Th. 6.7.

$$\{x \in X : f(x) = g(x)\} = \{x \in X : f(x) - g(x) = 0\} =$$

$$= \{x \in X : p(x) = 0\} = p^{-1}(\underbrace{\{0\}}_{\in \mathcal{B}(\mathbb{R})})$$

p is $\mathcal{F}$-measurable, then $p^{-1}(\{0\}) \in \mathcal{F}$ by defin.

Hence, $\{x \in X : f(x) = g(x)\} \in \mathcal{F}$.

(5) Let for a function $f: \mathbb{R} \to \mathbb{R}$ there exists the derivative f' on $\mathbb{R}$. Prove that f' is a Borel function.
[3p]

▶ A function $f: \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$.
It means that for $\forall x \in \mathbb{R}$ $\exists f'(x)$ and

$$f'(x) \overset{\text{defin. of derivative}}{=} \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Since f is differentiable, then f is continuous.
We know that every continuous function is Borel measurable (by application of corollary 6.2).
Take just the sequence

$$f_n(x) = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} = n\Big(\underbrace{f(x+\tfrac{1}{n})}_{\in B(\mathbb{R})} - \underbrace{f(x)}_{\in B(\mathbb{R})}\Big) \in B(\mathbb{R}). \text{ by Th. 6.7}$$

f_n is Borel measurable and f_n converges to f', $n \to \infty$.
By Th. 6.8 f' is also a Borel function.

◀

(6) [3p] Let $f_1, f_2 : X \to \mathbb{R}$ be non-negative simple functions such that $f_1(x) \le f_2(x)$, $x \in X$. Using the definition of the Lebesgue integral show that

$$\int_A f_1 \, d\lambda \le \int_A f_2 \, d\lambda \quad \text{for all } A \in \mathcal{F}.$$

▶ The function f_1, f_2 are simple. Then exist distinct $a_1, a_2, \dots, a_m \in \mathbb{R}$ and $b_1, b_2, \dots, b_n \in \mathbb{R}$ s.t.

$$f_1(x) = \sum_{k=1}^{m} a_k \, \mathbb{I}_{A_k}(x), \quad A_k \cap A_j = \emptyset, \ k \ne j, \quad \bigcup_{k=1}^{m} A_k = X$$

$$f_2(x) = \sum_{l=1}^{n} b_l \, \mathbb{I}_{B_l}(x), \quad B_l \cap B_i = \emptyset, \ l \ne i, \quad \bigcup_{l=1}^{n} B_l = X$$

By the definition of the Lebesgue intergal

$$\int_A f_1 d\lambda = \sum_{k=1}^{m} a_k \lambda(A_k \cap A) \ ; \quad \int_A f_2 d\lambda = \sum_{l=1}^{n} b_l \lambda(B_l \cap A).$$

Consider

$$\int_A f_1 d\lambda = \sum_{k=1}^{m} a_k \lambda(A_k \cap A) = \sum_{k=1}^{m} a_k \lambda\Big(A_k \cap A \cap \big(\underbrace{\bigcup_{l=1}^{n} B_l}_{=X}\big)\Big) =$$

$$= \sum_{k=1}^{m} a_k \lambda\Big(\bigcup_{l=1}^{n} (A_k \cap A \cap B_l)\Big) = \sum_{k=1}^{m} \sum_{l=1}^{n} a_k \lambda(A_k \cap A \cap B_l) \ \le$$

because $(A_k \cap A \cap B_i) \cap (A_k \cap A \cap B_l) = \emptyset$

Remart that for all $x \in A_k \cap B_l$

$$a_k = f_1(x) \le f_2(x) = b_l \quad \Rightarrow a_k \le b_l.$$

Consequently we can estimate the sum as follows

$$\le \sum_{k=1}^{m} \sum_{l=1}^{n} b_l \lambda(A_k \cap A \cap B_l) = \sum_{l=1}^{n} b_l \lambda\Big(\bigcup_{k=1}^{m} (A_k \cap A \cap B_l)\Big) =$$

$(A_k \cap A \cap B_l) \cap (A_j \cap A \cap B_l) = \emptyset$

$$= \sum_{l=1}^{n} b_l \lambda\Big(B_l \cap A \cap \big(\underbrace{\bigcup_{k=1}^{m} A_k}_{=X}\big)\Big) = \sum_{l=1}^{n} b_l \lambda(B_l \cap A) = \int_A f_2 d\lambda.$$

Hence, $\int_A f_1 d\lambda \le \int_A f_2 d\lambda$.

7 Let functions $f, g : \mathbb{N} \to \mathbb{R}$ satisfy

$$f^{-1}(\mathcal{B}(\mathbb{R})) \subset g^{-1}(\mathcal{B}(\mathbb{R})). \qquad (*)$$

Show that there exists a Borel function $F : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = F(g(x)), \quad x \in \mathbb{N}.$$

▶ Let $Im\, g := \{g(k) : k \in \mathbb{N}\}$.

Remark that $Im\, g$ is at most of countable.

Take for any $x \in Im\, g$

$$A_x := g^{-1}(\{x\}) = \{k : g(k) = x\} \neq \emptyset.$$

Let also $k_x = \min A_x$.

the first show that $\forall k, l \in A_x$ (it means that $g(l) = g(k)$)

it follows that

$$f(k) = f(l).$$

If it is not so, then take

$$A_1 := f^{-1}(\{f(k)\}), \quad A_2 := f^{-1}(\{f(l)\})$$

we have that $k \in A_1$, $l \in A_2$, $A_1 \cap A_2 = \emptyset$

since $A_1 \cap A_2 = f^{-1}(\{f(k)\} \cap \{f(l)\}) = f^{-1}(\emptyset) = \emptyset$.

But then $A_1, A_2 \in g^{-1}(\mathcal{B}(\mathbb{R}))$ by $(*)$ and hence,

$\emptyset \neq A_1 \cap A_x \in g^{-1}(\mathcal{B}(\mathbb{R}))$.

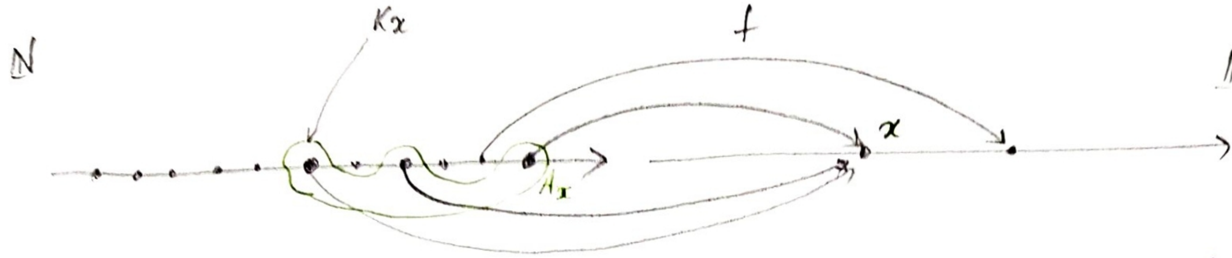
because $k \in A_1 \cap A_x$

So, $A_1 \cap A_x = g^{-1}(\tilde{A})$, $\tilde{A} \in \mathcal{B}(\mathbb{R})$.

Since $A_1 \cap A_x \subseteq A_x \Rightarrow \tilde{A} \subseteq \{x\}$.

So, $\tilde{A} = \{x\} \Rightarrow A_1 \cap A_x = A_x$. But this contradicts the fact that $l \notin A_1$. Thus $f(k) = f(l)$.

Set
$$F(x) = \begin{cases} f(k_x), & x \in Im\, g; \\ 0, & x \notin Im\, g. \end{cases}$$



Since $Im\,g$ is at most countable, any its subset is Borel measurable.

So, 1) $\forall\, C \geq 0$

$\{x : F(x) < C\} = \mathbb{R} \setminus \underbrace{\{x : f(x_k) \geq C\}}_{\subseteq Im\,g}$ – Borel measurable.

2) $\forall\, C < 0$

$\{x : F(x) < C\} = \{x : f(x_k) < C\} \subseteq Im\,g$ – Bolel measurable.

Let us show that

$$F(g(\ell)) = f(\ell).$$

Take $\ell \in \mathbb{N} \Rightarrow \ell \in A_x$, where $x = g(\ell)$.

So, $F(g(\ell)) = F(g(k_x)) = f(k_x) = f(\ell)$. ◂

(8) Let $X = \mathbb{N}$, $\mathcal{F} = 2^{\mathbb{N}}$ and $\lambda(\emptyset) := 0$, $\lambda(A) := \sum\limits_{n \in A} \frac{1}{n}$, $A \in 2^{\mathbb{N}}$.
Show that
a) $f \in L(\mathbb{N}, \lambda)$ if and only if $\sum\limits_{n=1}^{\infty} \frac{|f(n)|}{n} < +\infty$.
b) $\int\limits_{A} f \, d\lambda = \sum\limits_{n=1}^{\infty} \frac{f(n)}{n}$ for $f \in L(\mathbb{N}, \lambda)$.

a) Show that for any non-negative function f
[2p]
$$\int\limits_{\mathbb{N}} f \, d\lambda = \sum_{n=1}^{\infty} \frac{f(n)}{n}.$$

1) Let f is a simple function. Then
$$f(n) \underset{\text{by defin. of the simple func.}}{=} \sum_{k=1}^{m} a_k \, \mathbb{I}_{A_k}(n), \quad n \in \mathbb{N}$$

and
$$\int\limits_{\mathbb{N}} f \, d\lambda = \sum_{k=1}^{m} a_k \, \lambda(\underbrace{A_k \cap \mathbb{N}}_{A_k}) = \sum_{k=1}^{m} a_k \underbrace{\lambda(A_k)}_{= \sum\limits_{n \in A_k} \frac{1}{n} \text{ (given)}} =$$
$$= \sum_{k=1}^{m} a_k \sum_{n \in A_k} \frac{1}{n} = \sum_{k=1}^{m} \sum_{n \in A_k} \underbrace{a_k}_{= f(n) \text{ if } n \in A_k} \cdot \frac{1}{n} =$$
$$= \sum_{k=1}^{n} \sum_{n \in A_k} f(n) \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{f(n)}{n}.$$

2) Let f is any non-negative function. Then
$$\int\limits_{\mathbb{N}} f \, d\lambda = \sup_{p \in K(f)} \int\limits_{\mathbb{N}} p \, d\lambda,$$
$K(f)$ is the set of all simple function $p: \mathbb{N} \to \mathbb{R}$ s.t. $0 \le p(n) \le f(n)$, $\forall n \in \mathbb{N}$.

p is a simple function. By 1) $\int\limits_{\mathbb{N}} p \, d\lambda = \sum\limits_{n=1}^{\infty} \frac{p(n)}{n}$. Then
$$\int\limits_{\mathbb{N}} f \, d\lambda = \sup_{p \in K(f)} \int\limits_{\mathbb{N}} p \, d\lambda = \sup_{p \in K(f)} \sum_{n=1}^{\infty} \frac{p(n)}{n}.$$

Show that $\sup\limits_{p \in K(f)} \sum\limits_{n=1}^{\infty} \frac{p(n)}{n} \overset{?}{=} \sum\limits_{n=1}^{\infty} \frac{f(n)}{n}$.

$$\int\limits_{\mathbb{N}} p \, d\lambda \overset{1)}{=} \sum_{n=1}^{\infty} \frac{p(n)}{n} \underset{0 \le p(n) \le f(n), \forall n}{\le} \sum_{n=1}^{\infty} \frac{f(n)}{n} \Rightarrow \sup_{p \in K(f)} \underbrace{\sum_{n=1}^{\infty} \frac{p(n)}{n}}_{\int\limits_{\mathbb{N}} p \, d\lambda} \le \sum_{n=1}^{\infty} \frac{f(n)}{n}.$$

In order to show the equality we need to take a sequance of simple functions p_m such that

$$\int_{\mathbb{N}} p_m \, d\lambda \longrightarrow \sum_{n=1}^{\infty} \frac{f(n)}{n}, \quad m \to \infty.$$

Take

$$p_m(n) = \begin{cases} f(n), & n \leq m \\ 0, & n > m. \end{cases}$$

$$\underbrace{\int_{\mathbb{N}} p_m \, d\lambda = \sum_{n=1}^{m} \frac{f(n)}{n}}_{= \sum_{n=1}^{m} \frac{p_m(n)}{n}} \xrightarrow[m\to\infty]{} \sum_{n=1}^{\infty} \frac{f(n)}{n}.$$

Hence, $\int_{\mathbb{N}} f \, d\lambda = \sup_{p \in K(f)} \int_{\mathbb{N}} p \, d\lambda = \sum_{n=1}^{\infty} \frac{f(n)}{n}$ for any non-negative function f.

Let $f \in L(\mathbb{N}, \lambda)$. We know that $f \in L(\mathbb{N}, \lambda)$ if and only if $|f| \in L(\mathbb{N}, \lambda)$ (property 9, Lect. 7)

$$\int_{\mathbb{N}} |f| \, d\lambda = \sum_{n=1}^{\infty} \frac{|f(n)|}{n}$$

$|f| \geq 0$, then by a)

Hence $f \in L(\mathbb{N}, \lambda)$ if and only if $\sum_{n=1}^{\infty} \frac{|f(n)|}{n} < +\infty$.

b) Show that for any function $f \in L(\mathbb{N}, \lambda)$
[2p]

$$\int_{\mathbb{N}} f \, d\lambda = \sum_{n=1}^{\infty} \frac{f(n)}{n}.$$

A function $f \in L(\mathbb{N}, \lambda)$ if $\int_{\mathbb{N}} f_+ \, d\lambda < +\infty$ and $\int_{\mathbb{N}} f_- \, d\lambda < +\infty$,

where $f_+(n) = \max\{f(n); 0\}$

$f_-(n) = -\min\{f(n); 0\}$, $n \in \mathbb{N}$.

$$\int_{\mathbb{N}} f_+ \, d\lambda \overset{a)}{=} \sum_{n=1}^{\infty} \frac{f_+(n)}{n} \quad \text{and} \quad \int_{\mathbb{N}} f_- \, d\lambda \overset{a)}{=} \sum_{n=1}^{\infty} \frac{f_-(n)}{n}$$

$f_+(n) \geq 0$ $\qquad$ $f_-(n) \geq 0$

$$\int_{\mathbb{N}} f \, d\lambda = \int_{\mathbb{N}} f_+ \, d\lambda - \int_{\mathbb{N}} f_- \, d\lambda = \sum_{n=1}^{\infty} \frac{f_+(n)}{n} - \sum_{n=1}^{\infty} \frac{f_-(n)}{n} = \sum_{n=1}^{\infty} \frac{f_+(n) - f_-(n)}{n} =$$

$$= \sum_{n=1}^{\infty} \frac{f(n)}{n}.$$

Hence, $\int_{\mathbb{N}} f \, d\lambda = \sum_{n=1}^{\infty} \frac{f(n)}{n}$ for any $f \in L(\mathbb{N}, \lambda)$.