(1) Let $X = \{0,1,2\}$ and $H = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$.
Is the funktion $\mu: H \to \mathbb{R}$ a measure on H,
where μ is defined as

a) $\mu(\emptyset) := 0$, $\mu(\{0\}) = -1$, $\mu(\{1\}) = 2$ and $\mu(\{0,1\}) = 1$;
b) $\mu(\emptyset) := 0$, $\mu(\{0\}) = 1$, $\mu(\{1\}) = 2$ and $\mu(\{0,1\}) = 3$;
c) $\mu(\emptyset) := 0$, $\mu(\{0\}) = 1$, $\mu(\{1\}) = 2$ and $\mu(\{0,1\}) = 1$?

▶ If μ is a measure on H then

1) $\mu(\emptyset) = 0$;
2) $\mu(A) \geq 0 \quad \forall A \in H$;
3) $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k), \quad \forall A_k \in H, \qquad A_k \cap A_j = \emptyset, k \neq j.$

a) $\mu(\{0\}) = -1 < 0 \Rightarrow \mu$ is not a measure.

b) Check the conditions 1) – 3).
1) and 2) are obvious
3) $\underset{3}{\underset{\parallel}{\mu(\{0,1\})}} = \mu(\{0\}) + \mu(\{1\}) = 1 + 2 = 3$ $\Rightarrow \mu$ is a measure.
It is enougt for 3)

c) 1) and 2) are obvious
3) $\underset{\underset{\sim}{1}}{\underset{\parallel}{\mu(\{0,1\})}} = \mu(\{0\}) + \mu(\{1\}) = 1 + 2 = \underset{\sim}{\underline{3}} \Rightarrow$

$\Rightarrow \mu(\{0,1\}) \neq \mu(\{0\}) + \mu(\{1\})$
μ is not a measure. ◀

② Show that a nonnegative, additive and
[3p.] continuous[1] below function μ on a ring H is a measure on H.

[1] A function μ defined on a ring H is called continuous below, if for every increasing family $\{A_n : n \geq 1\} \subset H$ one has $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

▸ We have that H is a ring,

1) $\mu(A) \geq 0$, $A \in H$;
2) $\mu(\bigcup_{K=1}^{n} A_K) = \sum_{K=1}^{n} \mu(A_K)$, $\forall A_K \in H$, $K = \overline{1,n}$, $A_K \cap A_j = \emptyset$, $K \neq j$;
3) for every increasing family $\{A_n : n \geq 1\} \subset H$
$\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

We need to show that μ is a measure on H.
Check that

a) $\mu(A) \geq 0$ $\forall A \in H$;
b) $\forall A_n \in H$, $n \geq 1$, $A_K \cap A_n = \emptyset$, $K \neq n$
$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

a) $\mu(A) \geq 0$ by 1). !

b) Let $B_n = \bigcup_{K=1}^{n} A_K$:

$B_1 = A_1$
$B_2 = A_1 \cup A_2$
$- - - - - -$
$B_n = A_1 \cup A_2 \cup \ldots \cup A_n$

Then $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \left(\bigcup_{K=1}^{n} A_K \right) = \bigcup_{n=1}^{\infty} A_n$

and

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \mu(\underbrace{\bigcup_{n=1}^{\infty} B_n}_{B_n \text{ is a increasing family}}) \overset{\text{by } 3)}{=} \lim_{n\to\infty} \mu(\overbrace{B_n}^{=\bigcup_{K=1}^{n} A_K}) = \lim_{n\to\infty} \mu(\bigcup_{K=1}^{n} A_K) \overset{2)}{=}$$

$$= \lim_{n\to\infty} \sum_{K=1}^{n} \mu(A_K) = \sum_{n=1}^{\infty} \mu(A_n). \ !$$

μ is a measure on H. ◀

③ Let μ be a measure on a σ-algebra $H \subset 2^X$ and $\mu(X)=1$. For a family of sets $\{A_n : n \geq 1\} \subset H$ satisfying $\mu(A_n)=1$, $n \geq 1$, show that $\mu(\bigcap_{n=1}^{\infty} A_n) = 1$.

▸ μ is a measure on H. Then
1) μ is monotone: $\forall A, B \in H$ s.t. $A \subset B \Rightarrow$
$\Rightarrow \mu(A) \leq \mu(B)$;
2) $\forall A, B \in H$ s.t. $A \subset B \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$
3) μ is σ-semiadditive:
$\forall A_1, A_2, \ldots \in H$ s.t. $\bigcup_{n=1}^{\infty} A_n \in H \Rightarrow$
$\Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

We know that
$$\bigcap_{n=1}^{\infty} A_n = X \setminus \left(\bigcup_{n=1}^{\infty} A_n^c\right)$$
Then
$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu\left(X \setminus \left(\bigcup_{n=1}^{\infty} A_n^c\right)\right) \overset{2),\ (\bigcup_{n=1}^{\infty} A_n^c \subset X)}{=} \mu(X) - \mu\left(\bigcup_{n=1}^{\infty} A_n^c\right) \overset{3)\ \sigma\text{-semiadd.}}{\geq}$$
$$\geq \mu(X) - \sum_{n=1}^{\infty} \mu(A_n^c).$$
Show that $\mu(A_n^c) = 0$. Indeed, $\mu(A_n^c) = \mu(X \setminus A_n) =$
$$= \underbrace{\mu(X)}_{1\text{(given)}} - \underbrace{\mu(A_n)}_{1\text{(given)}} = 0$$
We obtain that
$$1 = \mu(X) \overset{1)}{\geq} \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \geq \mu(X) - \sum_{n=1}^{\infty} \underbrace{\mu(A_n^c)}_{0} = 1.$$
Hence, $\mu(\bigcap_{n=1}^{\infty} A_n) = 1$. ▲

④ Let μ be a measure on an algebra $H \subset 2^X$ and $\mu(X) = 1$. Let a family of sets
[4p.] $\{A_1, \dots, A_n\} \subset H$ satisfy the following property

$$\mu(A_1) + \dots + \mu(A_n) > n-1.$$

Show that $\mu\left(\bigcap_{k=1}^{n} A_k\right) > 0$.

▲

$$\bigcap_{k=1}^{n} A_k = X \setminus \left(\bigcup_{k=1}^{n} A_k^c\right)$$

Then

$$\mu\left(\bigcap_{k=1}^{n} A_k\right) = \mu\left(X \setminus \left(\bigcup_{k=1}^{n} A_k^c\right)\right) \underset{\text{monot.}}{\overset{\bigcup_{k=1}^{n} A_k^c \subset X}{=}} \mu(X) - \mu\left(\bigcup_{k=1}^{n} A_k^c\right). \qquad (*)$$

Show that $\mu\left(\bigcup_{k=1}^{n} A_k^c\right) < 1$.

Indeed, $\mu\left(\bigcup_{k=1}^{n} A_k^c\right) \underset{\sigma\text{-add.}}{\leq} \sum_{k=1}^{n} \mu(A_k^c) = \sum_{k=1}^{n} \mu(X \setminus A_k) =$

$$= \sum_{k=1}^{n} \left(\mu(X) - \mu(A_k)\right) = \underbrace{\sum_{k=1}^{n} \overset{1}{\overset{\shortparallel}{\mu(X)}}}_{n} - \underbrace{\sum_{k=1}^{n} \mu(A_k)}_{> n-1 \text{ (given)}} <$$

$$-\sum \mu(A_k) < -(n-1)$$

$$< n - (n-1) = 1.$$

Then

$$\mu\left(\bigcap_{k=1}^{n} A_k\right) \overset{(*)}{=} \underset{1}{\underset{\shortparallel}{\mu(X)}} - \underbrace{\mu\left(\bigcup_{k=1}^{n} A_k^c\right)}_{<1} > 1 - 1 = 0,$$

$$-\mu(\;) > -1$$

$$\mu\left(\bigcap_{k=1}^{n} A_k\right) > 0 \,!$$

⑤ For a sequence $\{A_n : n \geq 1\}$ of subsets of X define

$$\underline{\lim}_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k, \quad \overline{\lim}_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Let μ be a measure on a σ-algebra $H \subset 2^X$ and $\{A_n : n \geq 1\} \subset H$.

a) Prove that $\mu(\underline{\lim}_{n\to\infty} A_n) \leq \underline{\lim}_{n\to\infty} \mu(A_n)$.

b) Let additionally $\mu(\bigcup_{k=1}^{\infty} A_k) < +\infty$. Show that $\mu(\overline{\lim}_{n\to\infty} A_n) \geq \overline{\lim}_{n\to\infty} \mu(A_n)$.

a) $\underline{\lim}_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \underbrace{\bigcap_{k=n}^{\infty} A_k}_{B_n}$.

Let $B_n = \bigcap_{k=n}^{\infty} A_k$. Then $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n \subseteq \ldots$ and

$\mu(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu(B_n)$ (by Th. 3.9 (continuity from below))

$$\mu(\underline{\lim}_{n\to\infty} A_n) = \mu(\bigcup_{n=1}^{\infty} B_n) \overset{Th.\,3.9}{=} \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(\bigcap_{k=n}^{\infty} A_k). \quad (*)$$

We know that

$\mu(\bigcap_{k=n}^{\infty} A_k) \underset{monot.}{\leq} \mu(A_k)$, $\forall k \geq n$, because for every $k \geq n$ $\bigcap_{k=n}^{\infty} A_k \subseteq A_k$.

Then $\mu(\bigcap_{k=n}^{\infty} A_k) \leq \inf_{k\geq n} \mu(A_k)$. $(**)$

$$\mu(\underline{\lim}_{n\to\infty} A_n) \overset{(*)}{=} \lim_{n\to\infty} \mu(\bigcap_{k=n}^{\infty} A_k) \overset{(**)}{\leq} \lim_{n\to\infty} \inf_{k\geq n} \mu(A_k) = \underline{\lim}_{n\to\infty} \mu(A_n).!$$

b) $\overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} A_k}_{B_n}$.

Let $B_n = \bigcup_{k=n}^{\infty} A_k$. Then $B_1 \supseteq B_2 \supseteq \dots$, $\mu(B_1) = \mu(\bigcup_{k=1}^{\infty} A_k) \leq$

$\leq \sum_{k=1}^{\infty} \mu(A_k) \underset{\text{given.}}{<} +\infty$ and $\mu(\bigcap_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu(B_n)$ (by Th 3.10)

$$\mu(\overline{\lim_{n\to\infty}} A_n) = \mu(\bigcap_{n=1}^{\infty} B_n) \overset{\text{Th 3.10.}}{=} \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(\bigcup_{k=n}^{\infty} A_k). \quad (*)$$

We know that

$\mu(\bigcup_{k=n}^{\infty} A_k) \geq \mu(A_k)$, $\forall k \geq n$, because $A_k \subseteq \bigcup_{n=n}^{\infty} A_k$, $\forall k \geq n$.

Then $\mu(\bigcup_{k=n}^{\infty} A_k) \geq \sup_{k\geq n} \mu(A_k)$. $(**)$

$$\mu(\overline{\lim_{n\to\infty}} A_n) \overset{(*)}{=} \lim_{n\to\infty} \mu(\bigcup_{k=n}^{\infty} A_k) \overset{(**)}{\geq} \lim_{n\to\infty} \sup_{k\geq n} \mu(A_k) = \overline{\lim_{n\to\infty}} \mu(A_n). \ !$$

(6) Show that a nonnegative, additive and
[3b.] σ-semiadditive funktion μ on a ring R is
a measure on R.

▶ We have that H is a ring and

1) $\mu(A) \geq 0, \ \forall A \in H$;

2) $\mu\left(\bigcup_{K=1}^{n} A_K\right) = \sum_{K=1}^{n} \mu(A_K), \ \forall A_K \in H, \ K = \overline{1,n}, \ A_K \cap A_j = \emptyset, \ K \neq j$;

3) μ is σ-semiadditive, that is,

$\forall A_1, A_2, \ldots \in H$ s.t. $\bigcup_{n=1}^{\infty} A_n \in H \Rightarrow$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

We need to show that μ is a measure on H.
Check that

a) $\mu(A) \geq 0, \ \forall A \in H$;

b) $\forall A_n \in H, \ n \geq 1, \ A_K \cap A_n = \emptyset, \ K \neq n,$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

a) $\mu(A) \geq 0$ by 1).

b) $\Rightarrow$) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \underset{\sigma\text{-semi-add}}{\overset{\text{by } 3)}{\leq}} \sum_{n=1}^{\infty} \mu(A_n).$

$\Leftarrow$) Show that $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty} \mu(A_n)$?

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \overset{\text{by mon.}}{\geq} \mu\left(\bigcup_{K=1}^{n} A_K\right) \underset{\text{add.}}{\overset{\text{by } 2)}{=}} \sum_{K=1}^{n} \underbrace{\mu(A_n)}_{\geq 0}, \quad \forall n.$$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \lim_{n \to \infty} \sum_{K=1}^{n} \mu(A_n) = \sum_{n=1}^{\infty} \mu(A_n). \ !$$

μ is a measure on H. ▲

(7) [4p.] Let μ be a measure on a σ-algebra $H \subset 2^X$ and $\mu(X)=1$. Show that for every sequence $\{A_n : n \geq 1\} \subset H$ the equality

$$\sum_{n=1}^{\infty} \mu(A_n) < +\infty$$

implies $\mu(\overline{\lim_{n\to\infty}} A_n) = 0$.

▸ We have that

1) μ is a measure on a σ-alg. H;
2) $\mu(X) = 1$;
3) $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$, $\forall A_n \in H$, $n \geq 1$.

We need to show that $\mu(\overline{\lim_{n\to\infty}} A_n) = 0$.

We know that

$$\overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} A_k}_{B_n}$$

Let $B_n = \bigcup_{k=n}^{\infty} A_k$. Then $\overline{\lim_{n\to\infty}} A_n \subseteq B_n$ because $\overline{\lim_{n\to\infty}} A_n = \bigcap_{n=1}^{\infty} B_n$.

$$0 \overset{\text{be def. of meas.}}{\leq} \mu(\overline{\lim_{n\to\infty}} A_n) \overset{\text{by monot.}}{\leq} \mu(B_n) = \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \overset{\sigma\text{-semiadd}}{\leq} \sum_{k=n}^{\infty} \mu(A_n) \to 0$$

because $\sum_{k=1}^{\infty} A_k$ converges.

Hence, $\mu(\overline{\lim_{n\to\infty}} A_n) = 0$. !