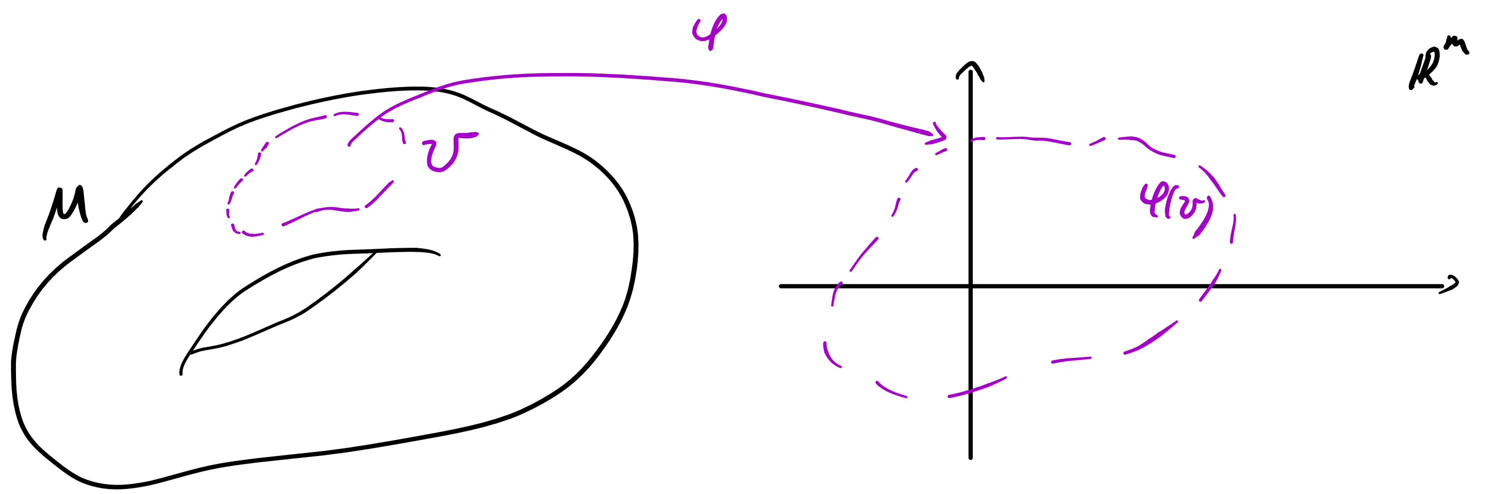
27. Differentiable Manifolds

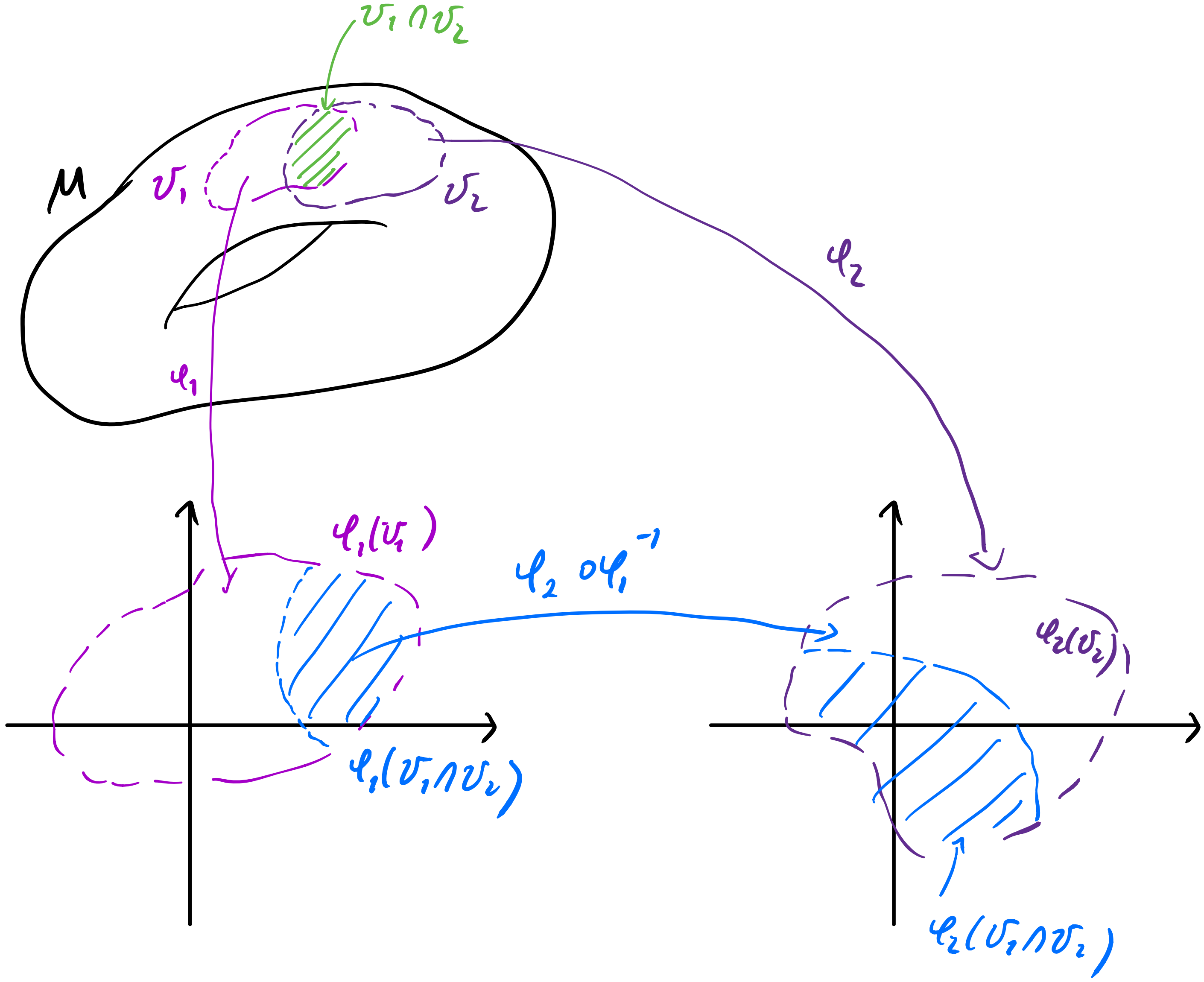
1. Main definitions

Assume that M is connected, hausdorf topological space. Connected means that there exist no open sets U, V such that $M = U \cup V$ and $U \cap V = \emptyset$.



Def 27.1 • An m-dimensional coordinate chart on M is a pair (U, φ) where U is an open subset of M (called the domain of coordinate chart) and φ is a homeomorphism of U onto an open subset of $\mathbb{R}^m$

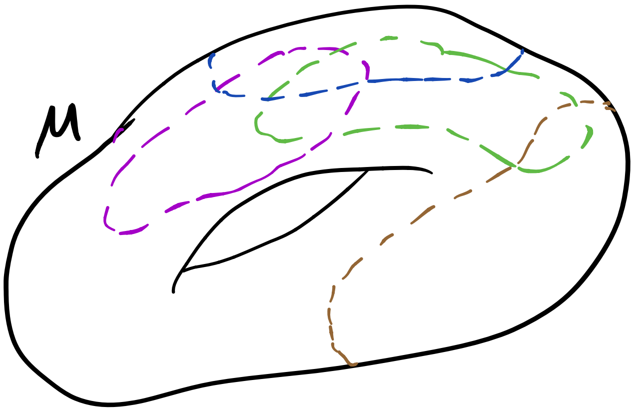
• If $U = M$ then the coordinate chart is globally defined, otherwise it is locally definite



Def 27.2 Let (U_1, φ_1), (U_2, φ_2) be m-dim. coordinate charts with $U_1 \cap U_2 \neq \emptyset$. Then the function

$$\varphi_2 \circ \varphi_1^{-1} : \underbrace{\varphi_1(U_1 \cap U_2)}_{\subset \mathbb{R}^m} \to \underbrace{\varphi_2(U_1 \cap U_2)}_{\subset \mathbb{R}^m}$$

is called the **overlap function**.



Def 27.3 • An atlas of dimension m on M is a family of m-dimensional coordinate charts $\{(U_i, \varphi_i)\}_{i\in I}$ (where I is an index set) such that

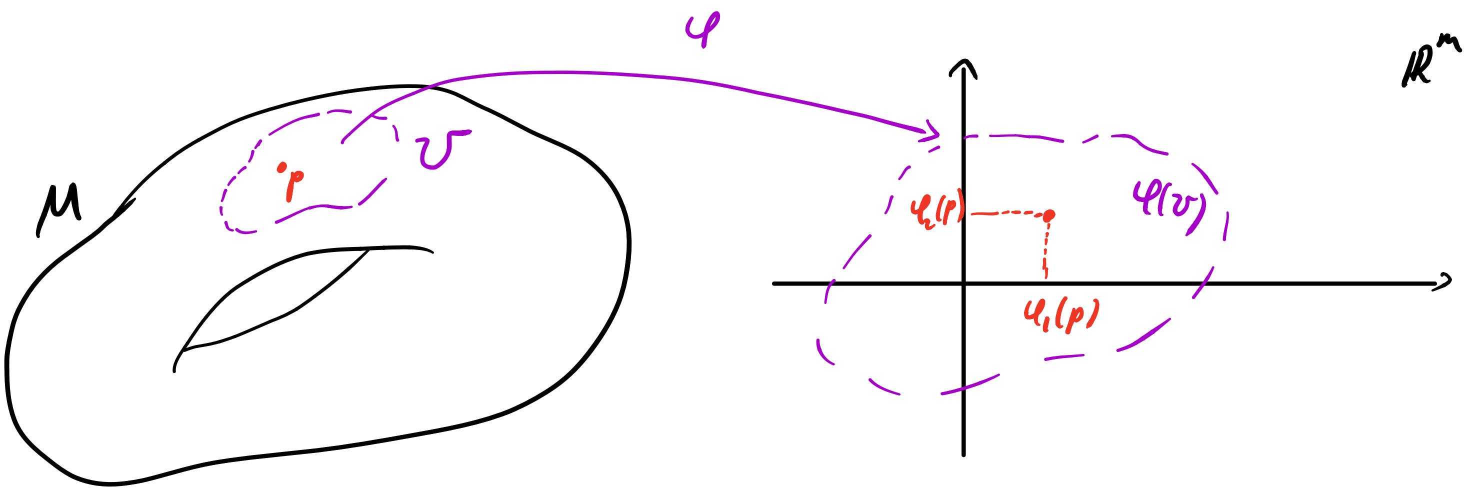
a) M is covered by $\{U_i\}_{i\in I}$, i.e.

$$M = \bigcup_{i\in I} U_i$$

b) each overlap function $\varphi_j \circ \varphi_i^{-1}$, $i,j\in I$ is infinitely differentiable (from class C^∞).

• An atlas is said to be complete if it is maximal, i.e. it is not contained in any other atlas

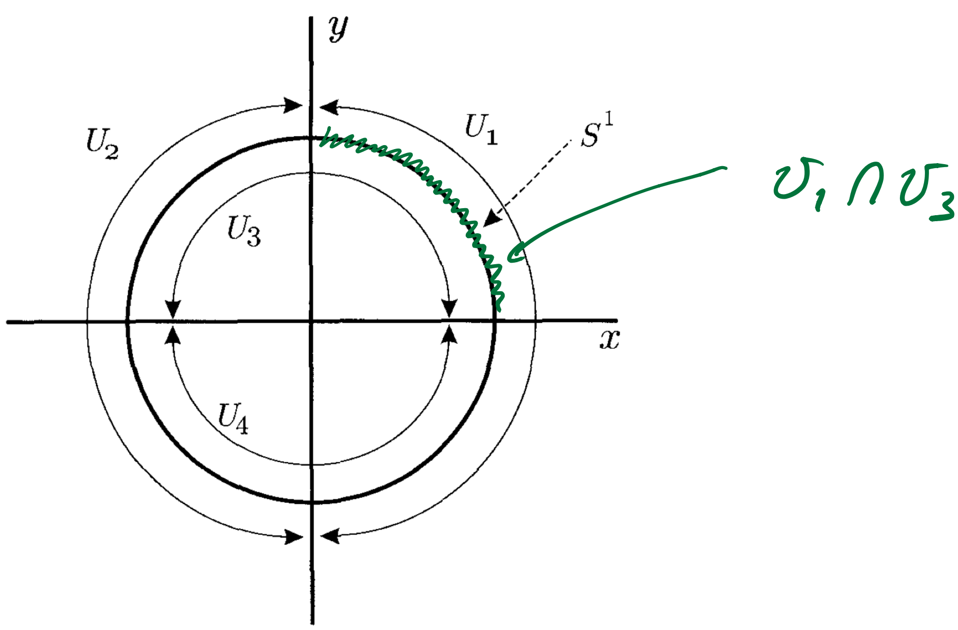
• For a complete atlas, the family $(U_i, \varphi_i)_{i\in I}$ is called a differential structure on M of dimension m. The topological space M is called then a differentiable manifold.



Def 27.4 A point $p \in U \subset M$ has the **coordinates** $(\varphi^1(p), \dots, \varphi^m(p))$ with respect to the chart (U, φ). The coordinates of p is often written as $(x^1(p), \dots, x^m(p))$.

2. Some examples of differentiable manifolds

a) The circle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$



One can define the differentiable structure on S^1 introducing the following charts

$$U_1 := \{(x,y) \in S^1 : x > 0\}, \quad \varphi_1(x,y) := y$$

$$U_2 := \{(x,y) \in S^1 : x < 0\}, \quad \varphi_2(x,y) = y$$

$$U_3 := \{(x,y) \in S^1 : y > 0\}, \quad \varphi_3(x,y) := x$$

$$U_4 := \{(x,y) \in S^1 : y < 0\}, \quad \varphi_4(x,y) := x$$

Let us show that the overlap functions are from C^∞. Consider the overlap of U_1 and U_3:

$$\varphi_1(x,y) = y$$

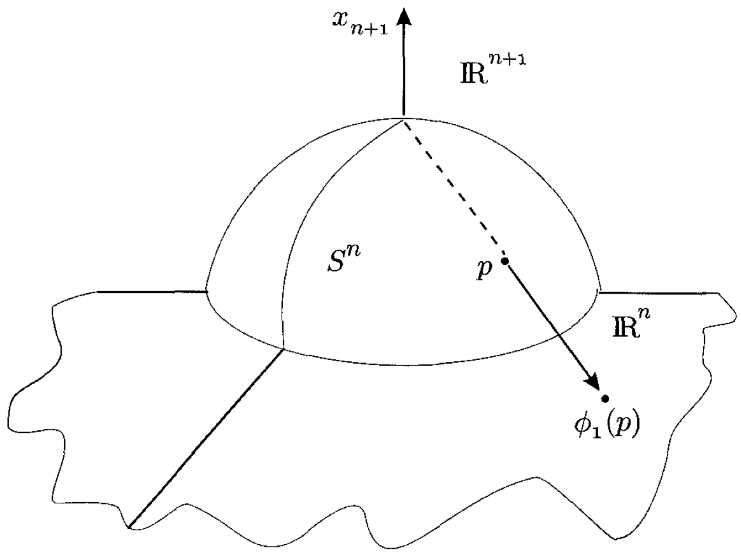
$$\varphi_3^{-1}(x) = \left(x, (1-x^2)^{\frac{1}{2}}\right).$$

Hence $$\varphi_1 \circ \varphi_3^{-1}(x) = (1-x^2)^{\frac{1}{2}}, \quad x \in (0,1)$$

- infinitely differentiable on $(0,1)$.

b) The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x|^2 = 1\}$

The differential structure can be given by means of stereographic projection from the north and south poles (φ_1 and φ_2, respectively)



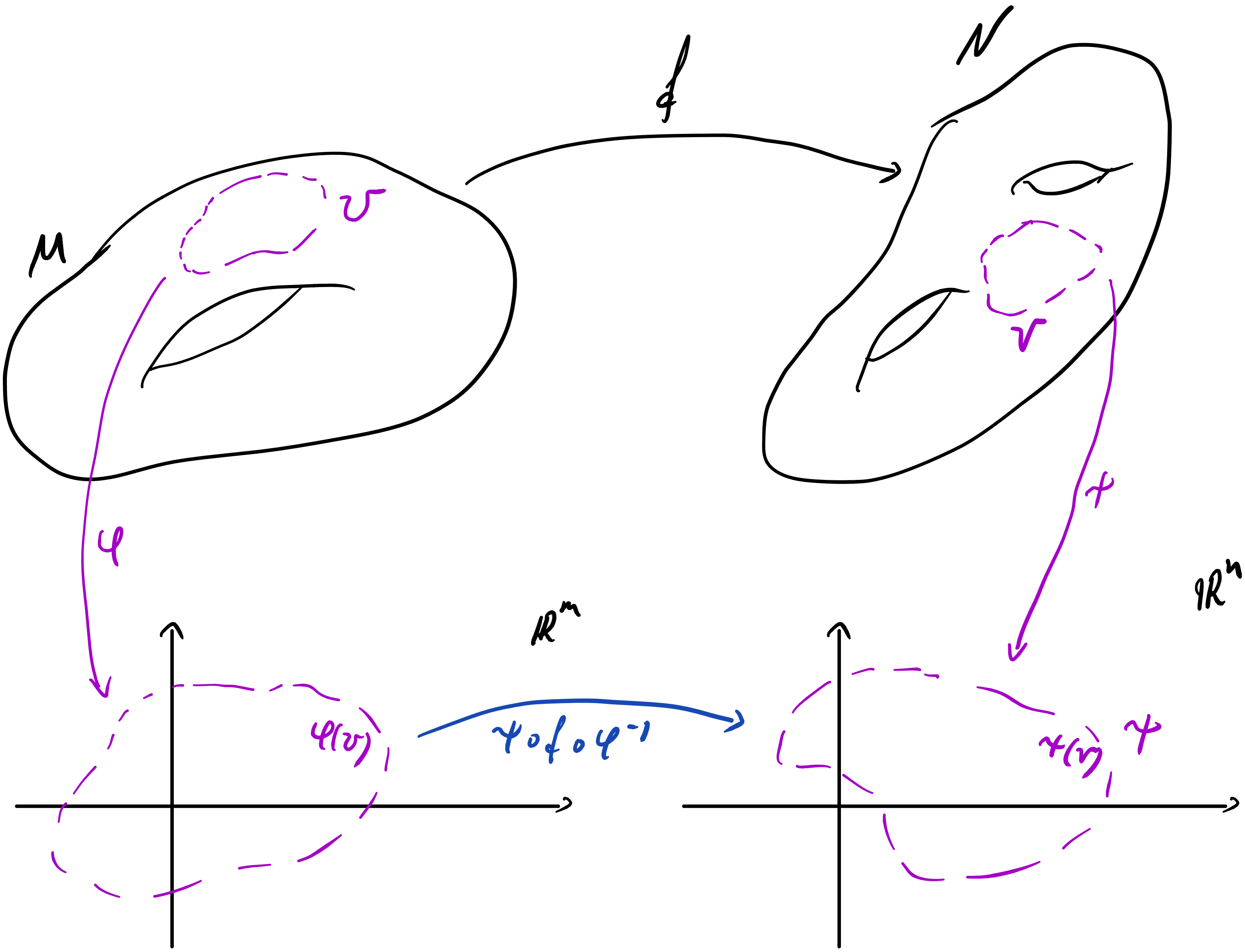
$$\varphi_1(x_1, \dots, x_{n+1}) := \left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}} \right) \in \mathbb{R}^n,$$

$$\varphi_2(x_1, \dots, x_{n+1}) := \left(\frac{x_1}{1+x_{n+1}}, \frac{x_2}{1+x_{n+1}}, \dots, \frac{x_n}{1+x_{n+1}} \right) \in \mathbb{R}^n.$$

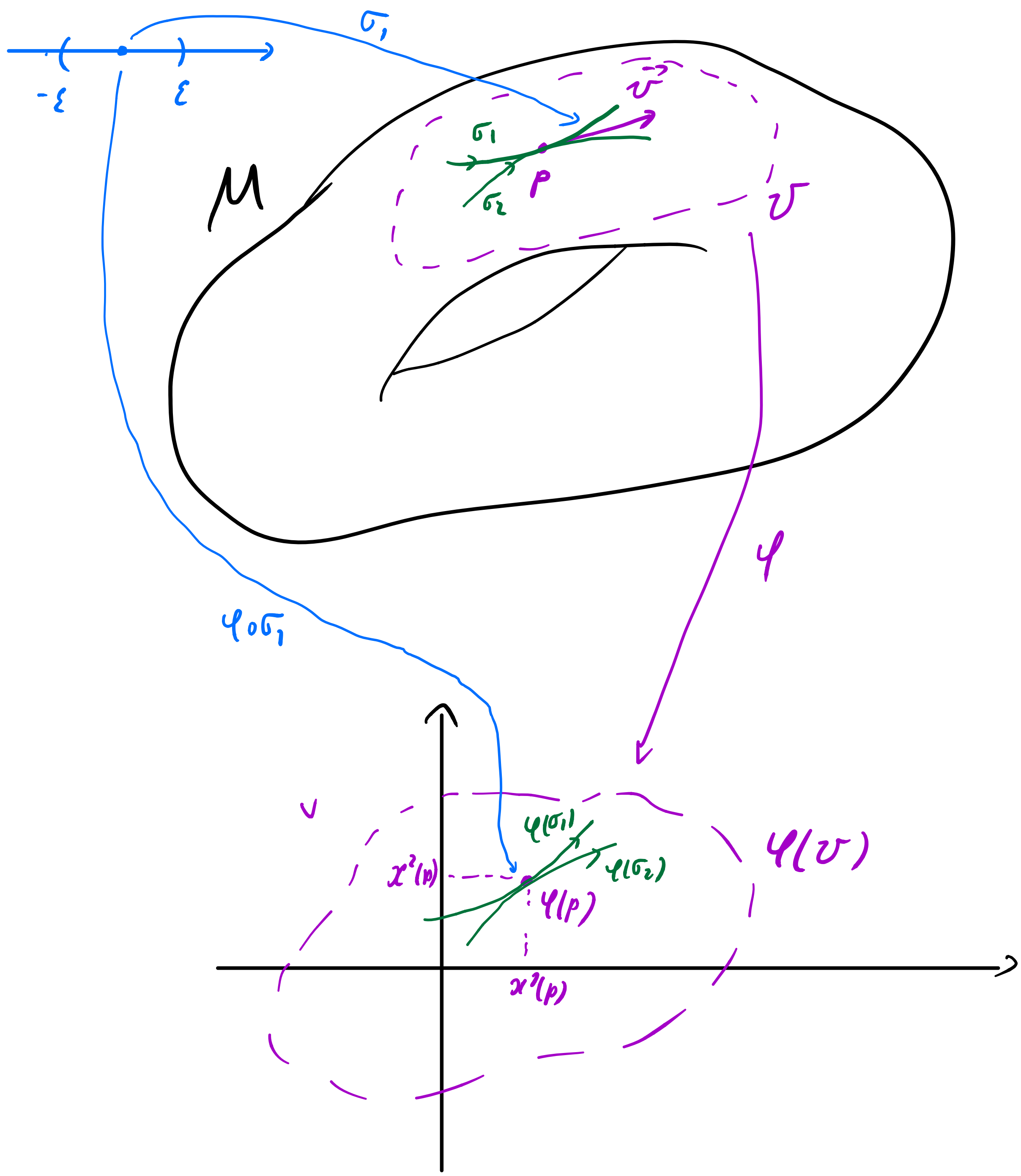
3. Differentiable maps and tangent space

Def 27.5. A *local representative* of a function $f: M \to N$ with respect to coordinate charts (U, φ) and (V, ψ) on M and N respectively, is the map

$$\psi \circ f \circ \varphi^{-1} : \underset{\mathbb{R}^m}{\underset{\cap}{\varphi(U)}} \to \mathbb{R}^n$$



- A map $f: M \to N$ is a C^r-function if for all coverings of M and N the local representatives are r times continuously differentiable. If f is C^1-funct. then f is called differentiable. If f is C^∞-function then f is called smooth.



Def 27.6 • A **curve** on a manifold M is a smoth map σ from some interval $(-\varepsilon, \varepsilon)$ of the real line into M.

• Two curves σ_1 and σ_2 are **tangent** at a point p in M if

1) $\sigma_1(0) = \sigma_2(0) = p$

2) in some local coordinate system $(x^1, \dots, x^m)$ around the point p

$$\frac{dx^i}{dt}(\sigma_1(t))\Big|_{t=0} = \frac{dx^i}{dt}(\sigma_2(t))\Big|_{t=0},$$

$i = 1, \dots, m$.

Remark that if σ_1 and σ_2 are tangent in one coordinate system, then they are tangent in any other coordinate system.

- A tangent vector at $p \in M$ is an equivalence class of tangent curves in p. The tangent class will be denoted by $[\sigma]$.

A tangent vector $v = [\sigma]$ can be used as a 'directional derivative' on functions $f\ M \to \mathbb{R}$ by defining

$$v(f) := \frac{d f(\sigma(t))}{dt}\Big|_{t=0},$$

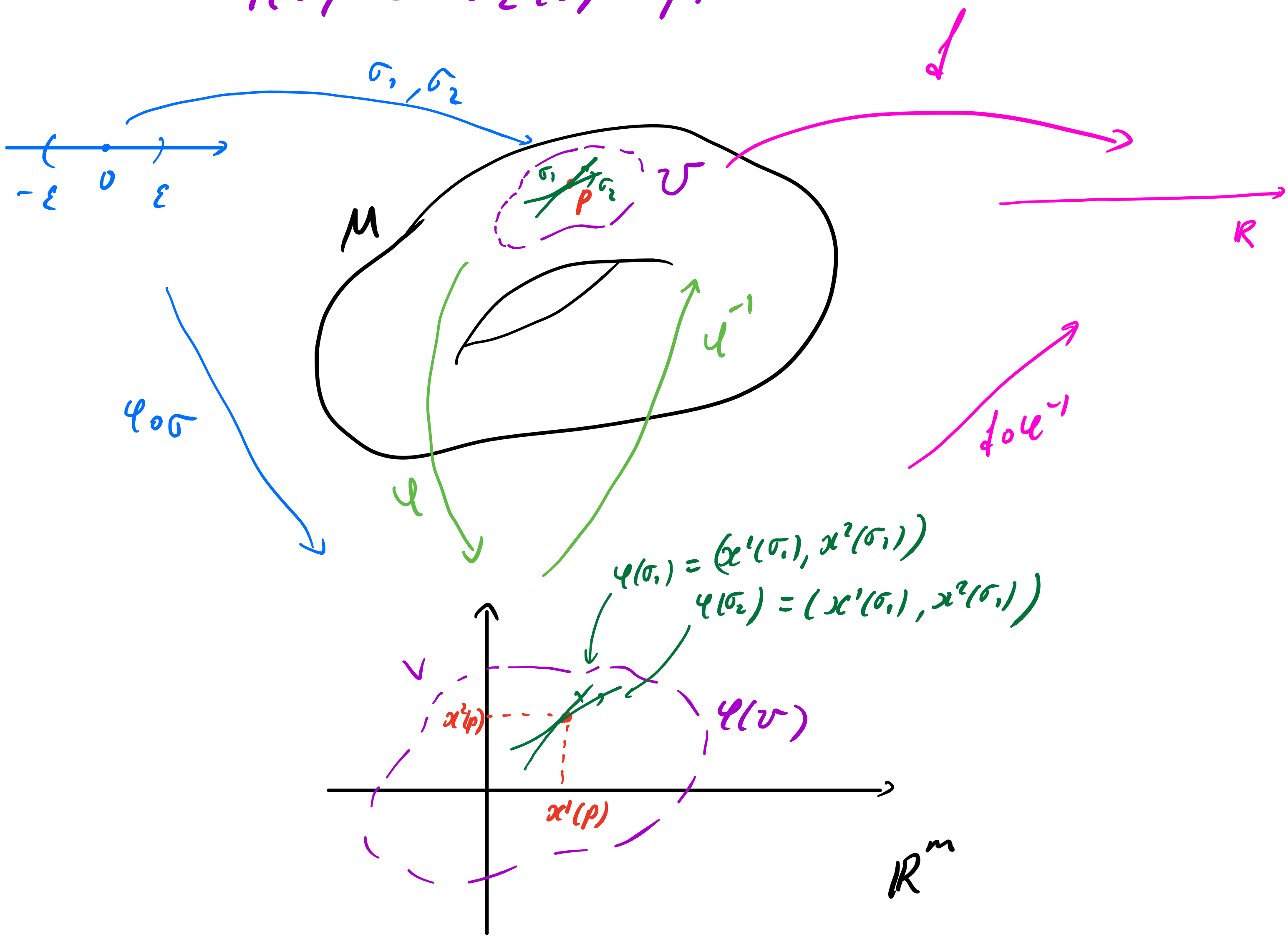
where σ is any curve from $[\sigma]$.

Remark that v does not depend on the choise of σ from $[\sigma]$.

Indeed, take any chart (U, φ) s.t. $p \in U$. Let σ_1, σ_2 be two curves such that

$$\frac{d x^i}{dt}(\sigma_1(t))\Big|_{t=0} = \frac{d x^i}{dt}(\sigma_2(t))\Big|_{t=0},$$

$$\sigma_1(0) = \sigma_2(0) = p.$$



$$\frac{d f(\sigma_1(t))}{dt}\Big|_{t=0} = \frac{d \ f \circ \varphi^{-1}(\varphi \circ \sigma_1)}{dt}\Big|_{t=0} =$$

$f \circ \varphi^{-1} : \underset{\mathbb{R}^m}{\varphi(U)} \to \mathbb{R}$

$$\varphi \circ \sigma : (-\varepsilon, \varepsilon) \to \mathbb{R}^m$$

$$= \sum_{i=1}^{m} \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} \cdot \frac{d\, x^i(\sigma_1)}{dt} \Big|_{t=0} =$$

$$= \sum_{i=1}^{m} \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} \cdot \frac{d\, x^i(\sigma_2)}{dt} \Big|_{t=0} =$$

$$= \frac{d f(\sigma_2(t))}{dt} \Big|_{t=0} .$$

Def 27.7 • The tangent space $T_p M$ to M at a point $p \in M$ is the set of all tangent vectors at the point p.

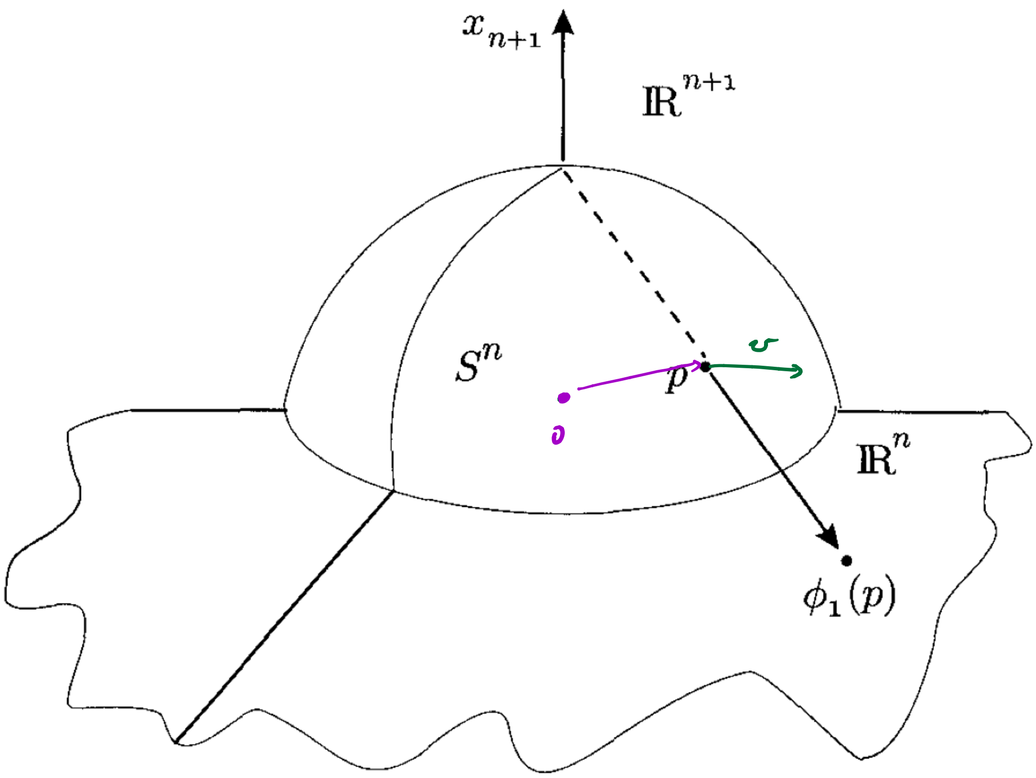
The tangent bundle TM is defined as

$$TM = \bigcup_{p \in M} T_p M .$$

Example 27.8 Let $M = S^n = \{ x \in \mathbb{R}^{n+1} : |x|^2 = 1 \}$

$$T_p S^n = \{ v \in \mathbb{R}^{n+1} : p \cdot v = 0 \}$$

$$TS^n = \{ (p, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |p|^2 = 1, \; p \cdot v = 0 \}$$



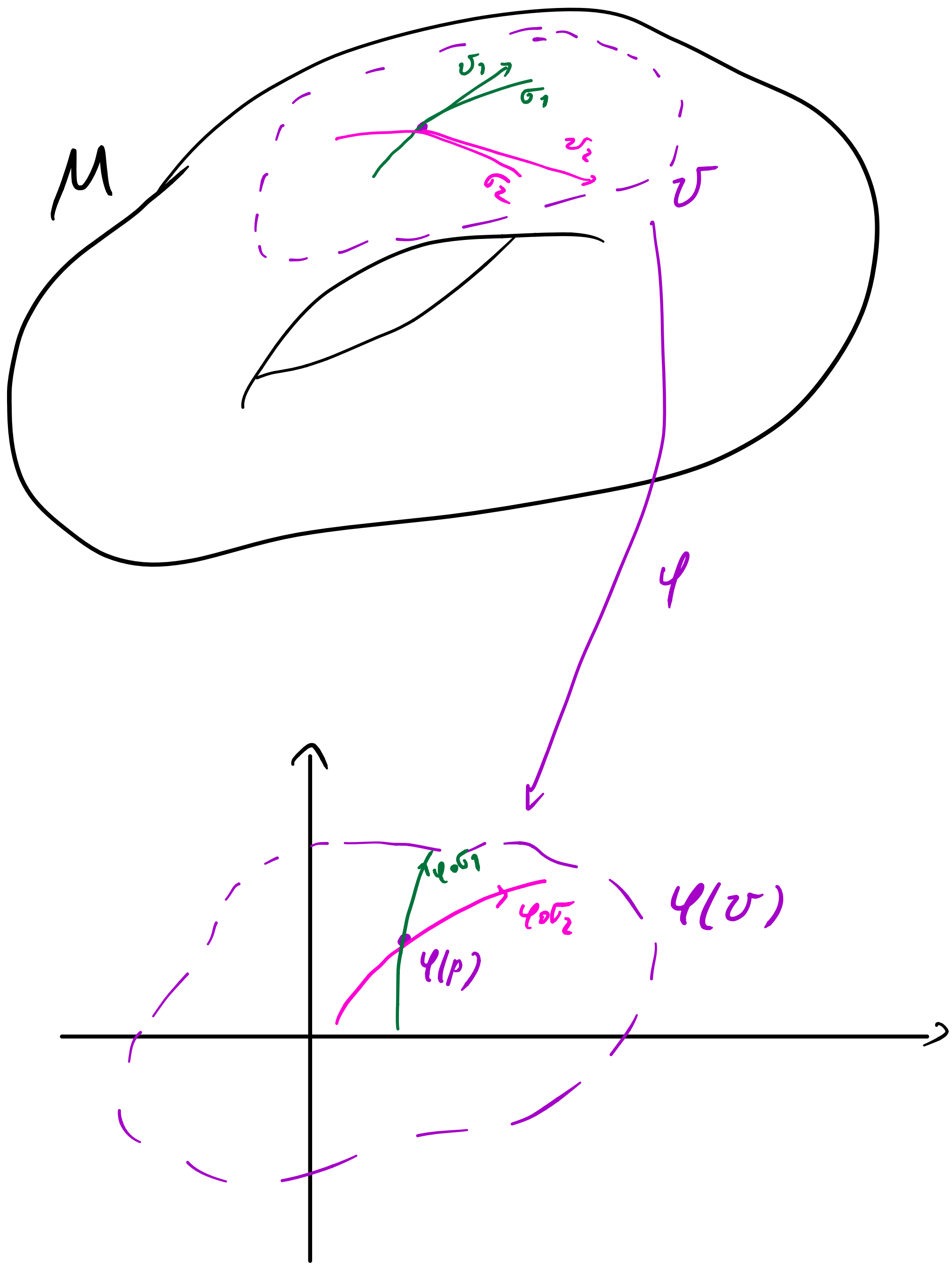
4. The vector space structure on T_pM.

The set T_pM can be make a vector space. Let v_1 and v_2 be two tangent vectors from T_pM. Let σ_1, σ_2 be two representative curves for v_1, v_2 respectively.

Of course, σ_1, σ_2 cannot be added directly since M is not a vector space. But we can consider the sum

$$t \mapsto \varphi \circ \sigma_1(t) + \varphi \circ \sigma_2(t)$$

which is a curve in $\mathbb{R}^m$.



So, we can define

$$v_1 + v_2 := [\varphi^{-1} \circ (\varphi\circ\sigma_1 + \varphi\circ\sigma_2)] \tag{27.1}$$

$$r v_1 := [\varphi^{-1} \circ (r \cdot \varphi\circ\sigma_1)], \quad r \in \mathbb{R}.$$

This definition are independent of

the choice of chart (U, φ) and representatives σ_1 and σ_2 of the tangent vectors v_1 and v_2.

Under the operations defined by (27.1) the set $T_p M$ is a vector space.

A tangent vector also can be defined as a derivation

$$v(f) = \left.\frac{d f(\sigma(t))}{dt}\right|_{t=0} \quad \text{where } [\sigma] = v.$$

Def 27.9 • A derivation at a point $p \in M$ is a map $v: C^\infty(M) \to \mathbb{R}$ such that

i) $v(f+g) = v(f) + v(g)$

$v(rf) = r\, v(f)$, $\quad r \in \mathbb{R}, f, g \in C^\infty(M)$

ii) $v(fg) = f(p) v(g) + g(p) v(f) \quad \forall f, g \in C^\infty(M)$.

• The set of all derivation is denoted by $D_p M$.

Th 27.10 The linear map $\iota : T_p M \to D_p M$ defined by

$$L(v)(f) := \frac{d f(\sigma(t))}{dt}\Big|_{t=0}, \quad [\sigma] = v$$

is an isomorphism.