26 Curves in $\mathbb{R}^3$. Topological spaces

1. Tangent, principal normal and binormal vectors. [Kreyszig, Diff. geom.]

Let

$$x : I \to \mathbb{R}^3,$$

where $x(t) = (x_1(t), x_2(t), x_3(t))$, $t \in I = [a,b]$ be a curve C. Let also

$$x = x(s)$$

be a natural parametrization of x, that is

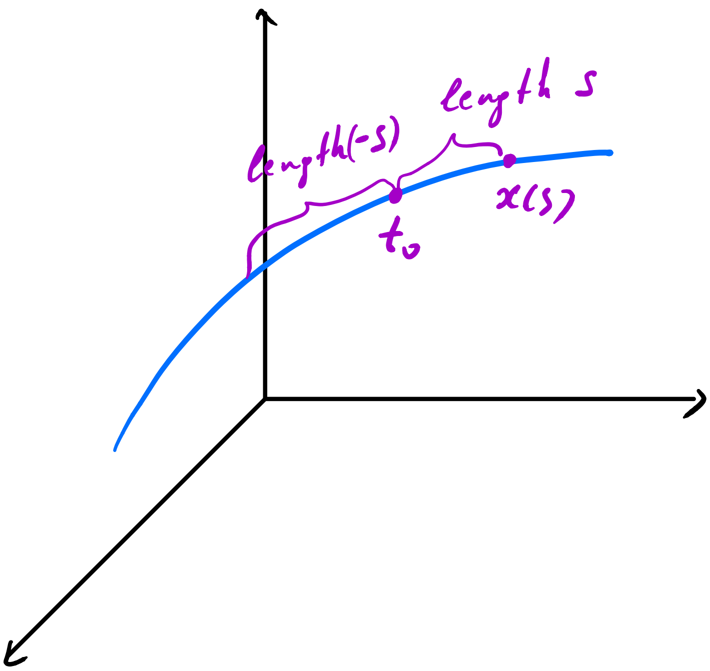
$$x(s) = x(t(s)),$$

where $t = t(s)$ is the inverse function to

$$s = \int_{t_0}^{t} |x'(t)|\, dt =$$

$$= \int_{t_0}^{t} \sqrt{(x_1'(u))^2 + (x_2'(u))^2 + (x_3'(u))^2}\, du.$$

Remark 26.1 $x = x(t)$ is the normal parametrisation ($s = t$) iff $|x'(t)| = 1$ for every $t \in I$.



Def 26.2 • The vector

$$\vec{t}(s) = \lim_{h \to 0} \frac{x(s+h) - x(s)}{h} = \frac{dx}{ds}(s) = \dot{x}(s)$$

is called the **unit tangent vector** to the curve C at the point $x(s)$.

If x is any (not normal) parametrization, then

$$\vec{t}(t) = \frac{x'(t)}{|x'(t)|}.$$

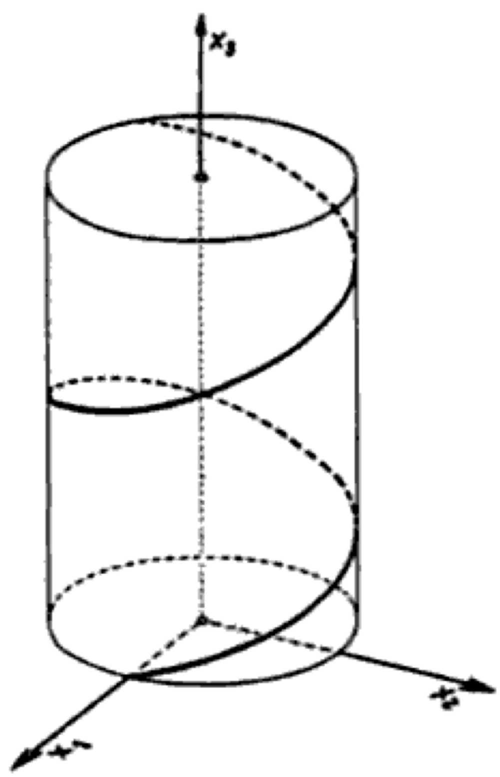
• The plain orthogonal to $\vec{t}(s)$ and passing through $x(s)$ is called the **normal plane**. It can be represented in the form

$$\dot{x}(s) \cdot z + x(s) = 0, \quad z = (z_1, z_2, z_3) \in \mathbb{R}^3.$$

Example 26.3 We consider the circular helix

$$x(t) = (r \cos t, \ r \sin t, \ ct), \quad t \in I$$

$c \neq 0$.



$$x'(t) = (-r\sin t,\ r\cos t,\ c)$$

$$|x'(t)| = \sqrt{r^2\sin^2 t + r^2\cos^2 t + c^2} =$$

$$= \sqrt{r^2+c^2}$$

$$s(t) = \int_0^t \sqrt{r^2+c^2}\,dt = \sqrt{r^2+c^2}\,t$$

$$t(s) = \frac{1}{\underbrace{\sqrt{r^2+c^2}}_{=:w}}\, s = \frac{s}{w}$$

$$x(s) = \left(r\cos\frac{s}{w},\ r\sin\frac{s}{w},\ \frac{c}{w}s\right)$$

$$\vec{t}(s) = \left(-\frac{r}{w}\sin\frac{s}{w},\ \frac{r}{w}\cos\frac{s}{w},\ \frac{c}{w}\right)$$

Def 26.4 . The rate of change of the tangent

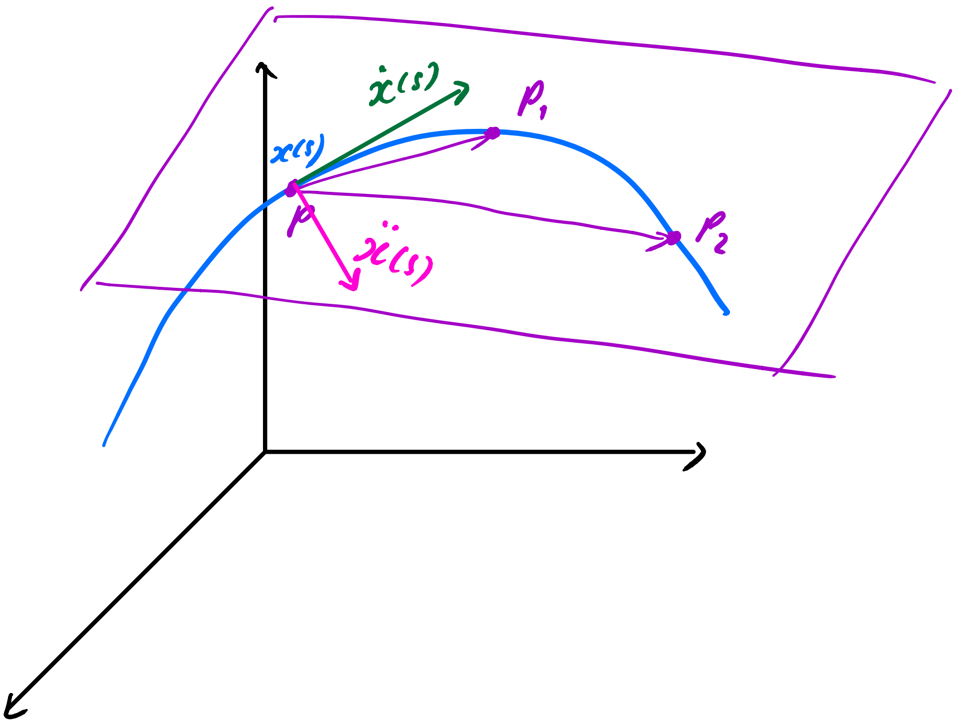
$$\kappa(s) = |\dot{\vec{t}}(s)| = |\ddot{x}(s)|$$

is called the **curvature** of the curve C at the point $x(s)$.

$$\kappa(t) = \frac{|x'(t)\times x''(t)|}{|x'(t)|^3}$$

- The plane passing through $x(s)$ and is parallel to $\dot{x}(s)$ and $\ddot{x}(s)$ (if $\ddot{x}(s) \neq 0$) is called the osculating plane

The plain can be obtained as the limit of the planes passing through $P P_1 P_2$ as $P_1, P_2 \to P$.



- The vector
$$\vec{p}(s) = \frac{\dot{\vec{t}}(s)}{|\dot{\vec{t}}(s)|} = \frac{\ddot{x}(s)}{|\ddot{x}(s)|} = \frac{1}{\kappa(s)} \ddot{x}(s)$$

is called the principal normal to the curve C at the point $x(s)$.

Example 26.3

$$x(s) = \left(r\cos\frac{s}{w},\ r\sin\frac{s}{w},\ \frac{c}{w}s \right)$$

$$\ddot{x}(s) = \left(-\frac{r}{w^2}\cos\frac{s}{w},\ -\frac{r}{w^2}\sin\frac{s}{w},\ 0 \right)$$

$$\kappa(s) = |\ddot{x}(s)| = \frac{r}{w^2} = \frac{r}{r^2+c^2}$$

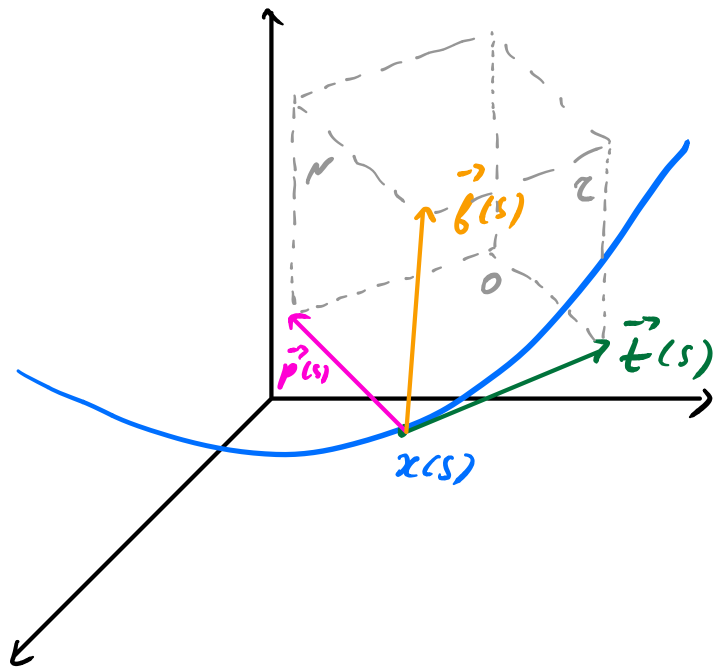
$$\vec{p}(s) = \left(-\cos\frac{s}{w},\ -\sin\frac{s}{w},\ 0 \right)$$

Def. 26.5. • The vector

$$\vec{b}(s) = \vec{t}(s) \times \vec{p}(s)$$

is called the binormal vector of C at the point $x(s)$.

• The plane parallel to $\vec{t}$ and $\vec{b}$ and passing through $x(s)$ is called the rectifying plane



Example 26.3

$$x(s) = \left(r\cos\frac{s}{w},\ r\sin\frac{s}{w},\ \frac{c}{w}s \right)$$

$$\vec{t}(s) = \left(-\frac{r}{w}\sin\frac{s}{w},\ \frac{r}{w}\cos\frac{s}{w},\ \frac{c}{w} \right)$$

$$\vec{p}(s) = \left(-\cos\frac{s}{w},\ -\sin\frac{s}{w},\ 0 \right)$$

$$\vec{b}(s) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{r}{w}\sin\frac{s}{w}, & \frac{r}{w}\cos\frac{s}{w} & \frac{c}{w} \\ -\cos\frac{s}{w}, & -\sin\frac{s}{w}, & 0 \end{vmatrix} =$$

$$= \left(\frac{c}{w}\sin\frac{s}{w},\ -\frac{c}{w}\cos\frac{s}{w},\ \frac{r}{w} \right)$$

We next introduce torsion. Roughly speaking it has to measure the rate of the rotation of curve, that is, the rate of change of the osculating plane. Assume that $\kappa(s) > 0$.

Def 26.6. The scalar

$$\tau(s) = -p(s) \cdot \dot{b}(s) = \frac{(\dot{x}(s), \ddot{x}(s), \dddot{x}(s))}{|\ddot{x}(s)|}$$

is called the **torsion** of the curve C at the point $x(s)$.

Example 26.3.

$$x(s) = \left(r \cos \frac{s}{w},\ r \sin \frac{s}{w},\ \frac{c}{w} s\right)$$

$$\vec{p}(s) = \left(-\cos \frac{s}{w},\ -\sin \frac{s}{w},\ 0\right)$$

$$\vec{b}(s) = \left(\frac{c}{w} \sin \frac{s}{w},\ -\frac{c}{w} \cos \frac{s}{w},\ \frac{r}{w}\right)$$

$$\dot{\vec{b}}(s) = \left(\frac{c}{w^2} \cos \frac{s}{w},\ \frac{c}{w^2} \sin \frac{s}{w},\ \frac{1}{w}\right)$$

$$\tau(s) = +\frac{c}{w^2} = \frac{c}{r^2 + c^2}.$$

Th 26.7. A curve (of class $C \geq 3$) with $\kappa(s) \neq 0$ $\forall s$ is a helix iff $\tau = \text{const}$, $\kappa = \text{const}$.

We remark that the vectors $\vec{t}, \vec{p}, \vec{b}$ form a basis. Consequently every vector can be rewritten as linear combination of this vectors. In particular we obtain

$$\begin{pmatrix} \dot{\vec{t}} \\ \dot{\vec{p}} \\ \dot{\vec{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \vec{t} \\ \vec{p} \\ \vec{b} \end{pmatrix}$$

– the Frenet formulae.

Th 26.8 A curve with $\kappa \neq 0$ belongs to a plane iff $\tau(s) = 0$ $\forall s$.

Th 26.9. If the curve C is given by an arbitrary parametrization, then

$$\vec{t}(t) = \frac{x'(t)}{|x'(t)|}, \quad \vec{p}(t) = \vec{b}(t) \times \vec{t}(t)$$

$$\vec{b}(t) = \frac{x'(t) \times x''(t)}{|x'(t) \times x''(t)|}$$

$$\kappa(t) = \frac{x'(t) \times x''(t)}{|x'(t)|^3}$$

$$\tau(t) = \frac{(x'(t), x''(t), x'''(t))}{|x'(t) \times x''(t)|^2}$$

For the proof of the theorem see p. 72 [O'Neill, Elem. diff. geom.].

2. Topological spaces

Let X be a set and $\mathcal{T}$ be a class of subsets which satisfies the following properties

(T1) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$

(T2) Any arbitrary (finite or infinite) union of sets from $\mathcal{T}$ belongs to $\mathcal{T}$

(T3) The intersection of a finitely many sets from $\mathcal{T}$ belongs to $\mathcal{T}$.

Def 26.9 The pair $(X, \mathcal{T})$ is called a **topological space**, where $\mathcal{T}$ satisfies (T1)–(T3). Sets from $\mathcal{T}$ are called **open sets**

Example 26.10 a) Let X be a metric space and $\mathcal{T}$ be a family of all open subsets from X. Then X is a topological space.

b) Take $X = [0,1]$ and

$$\mathcal{T} = \{ [0,b) : b \in (0,1) \} \cup \{\emptyset, X\}$$

Then X is also a topological space.

Def 26.11 A topological space $(X, \mathcal{T})$ is called **Hausdorff** if $\forall\, x, y \in X$ $\exists\, A, B \in \mathcal{T}$ s.t. $A \cap B = \emptyset$ and $x \in A$, $y \in B$.

Def 26.12 Let $(X, \mathcal{T})$, $(X', \mathcal{T}')$ be topological spaces. A function

$$f : X \to X'$$

is continuous if $\forall A \in \mathcal{T}'$

$$f^{-1}(A) \in \mathcal{T}.$$

Remark 26.13 If X, X' are metric spaces, then $f: X \to X'$ is continuous (as a function between metric spaces) iff it is continuous according to Def. 26.12.

Def 26.14 A map $f: X \to X'$ is called a **homeomorphism** if

a) f is bijection;

b) f and f^{-1} are continuous.

Let us consider a way of construction of topology. Assume that $\mathcal{B}$ is a collection of subsets from X such that

a) $\mathcal{B}$ covers X

b) $\forall B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ $\exists B_3 \in \mathcal{B}$ s.t. $B_3 \subset B_1 \cap B_2$.

Then the collection of arbitrary (finite or infinite) unions of subsets

from $\mathcal{B}$ is a topology on X.
This topology is called the topology generated by $\mathcal{B}$ and $\mathcal{B}$ is called the base of this topology.