25. Spectral representation of unbounded self-adjoint operators. Curves in $\mathbb{R}^3$.

1. Spectral representation

Let H be a complex Hilbert space. We recall that a bounded operator $U: H \to H$ is called unitary if

$$U^* = U^{-1}.$$

Th 25.1 Let $U: H \to H$ be a unitary operator. Then there exists a spectral family $\{E_\theta\}$ on $[-\pi, \pi]$ such that

$$U = \int_{-\pi}^{\pi} e^{i\theta} \, d E_\theta . \qquad (25.1)$$

(where the integral is understood in the sence of uniform operator convergence)

Proof (Idee) One can show that there exists a bounded self-adjoint linear operator S with $\sigma(S) \subset [-\pi, \pi]$ such that

$$U = e^{iS} = \cos S + i \sin S$$

Let $\{E_\theta\}$ be a spectral family for S on $[-\pi, \pi]$. Then

$$S = \int_{-\pi}^{\pi} \theta \, dE_\theta$$

Hence,

$$U = e^{iS} = \int_{-\pi}^{\pi} \cos\theta \, dE_\theta + i \int_{-\pi}^{\pi} \sin\theta \, dE_\theta$$

$$= \int_{-\pi}^{\pi} e^{i\theta} dE_\theta .$$

$\blacksquare$

Let $T : D(T) \to H$ be a self-adjoint linear operator, where $D(T)$ is dence in H and T may be unbounded.

We take a new operator

$$U = (T - iI)(T + iI)^{-1}$$

this operator is called the Cayley transform of T. It is defined on the whole Hilbert space since $-i \notin \sigma(T) \subseteq \mathbb{R}$. One can also check that it is unitary and

$$T = i(I + U)(I - U)^{-1}$$

Th 25.2 (Spectral representation for unbounded self-adjoint operators)
Let $T : \mathcal{D}(T) \to H$ be a self-adjoint linear-operator and $\mathcal{D}(T)$ is dense in H. Let $\mathcal{V}$ be the Cayley transform of T and $\{\tilde{E}_\theta\}$ be a spectral family in the spectral representation (25.1) for $-\mathcal{V}$. Then

$$T = \int_{-\pi}^{\pi} \tan\frac{\theta}{2}\, d\tilde{E}_\theta =$$

$$= \int_{-\infty}^{+\infty} \lambda \, dE_\lambda ,$$

where $E_\lambda = \tilde{E}_{2\arctan\lambda}$, $\lambda \in \mathbb{R}$

Motivation for the form of the representation.

We remark that

$$T = i(I+\mathcal{V})(I-\mathcal{V})^{-1} =$$

$$= f(-\mathcal{V}),$$

where $f(\theta) = i\,\dfrac{1-\theta}{1+\theta}$

Let

$$-V = \int_{-\pi}^{\pi} e^{i\theta} \, d\tilde{E}_\theta .$$

Then

$$T = \int_{-\pi}^{\pi} f(-e^{i\theta}) \, d\tilde{E}_\theta =$$

$$= \int_{-\pi}^{\pi} i \frac{1 - e^{i\theta}}{1 + e^{i\theta}} \, d\tilde{E}_\theta =$$

$$= \int_{-\pi}^{\pi} i \frac{(1-\cos\theta) - i\sin\theta}{(1+\cos\theta) + i\sin\theta} \, d\tilde{E}_\theta = \ldots =$$

$$= \int_{-\pi}^{\pi} i \frac{-2i\sin\theta}{2 + 2\cos\theta} \, d\tilde{E}_\theta = \int_{-\pi}^{\pi} \tan\frac{\theta}{2} \, d\tilde{E}_\theta .$$

Example 25.3 (Spectral representation of the multiplication operator)

Let $H = L^2(-\infty, +\infty)$ be taken over $\mathbb{C}$, and

$$(Tx)(t) = t\,x(t), \quad t \in \mathbb{R},$$

$$\mathcal{D}(T) = \left\{ x \in L^2(-\infty, +\infty) : \int_{-\infty}^{+\infty} t^2 |x(t)|^2 \, dt < +\infty \right\}$$

Then T is self-adjoint and

$$(F_\lambda x)(t) = \begin{cases} x(t), & t < \lambda \\ 0, & t \geq \lambda \end{cases}$$

is the spectral family associated with T.

III Differential geometry

Curves in $\mathbb{R}^3$

2. Some definitions

We consider a map

$$x : I \to \mathbb{R}^3,$$

where $x(t) = (x_1(t), x_2(t), x_3(t))$, $t \in I$, $I = [a, b]$. We assume that

- x_i are r times continuously differentiable and
- for every $t \in I$,

$$x'(t) = (x_1'(t), x_2'(t), x_3'(t)) \neq 0.$$

A set of points represented by x we will call a curve. A curve can have different representation. Indeed, let us consider a transformation

$$t = t(t^*) \qquad (25.2)$$

such that

1) $t : [a^*, b^*] \to [a, b]$, $t(a^*) = a$, $t(b^*) = b$
(or $t(a^*) = b$, $t(b^*) = a$)

2) the function is r time continuously differendiable

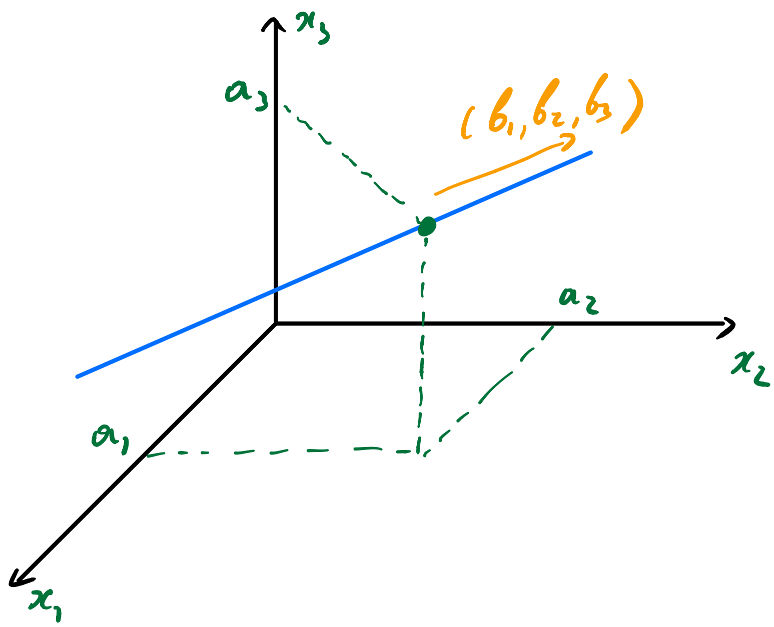
3) $\frac{dt}{dt^*}$ is different from zero on I^*.

Then $x(t(t^*)) =: x(t^*)$ is another parametrization of the curve x.

Examples 25.4

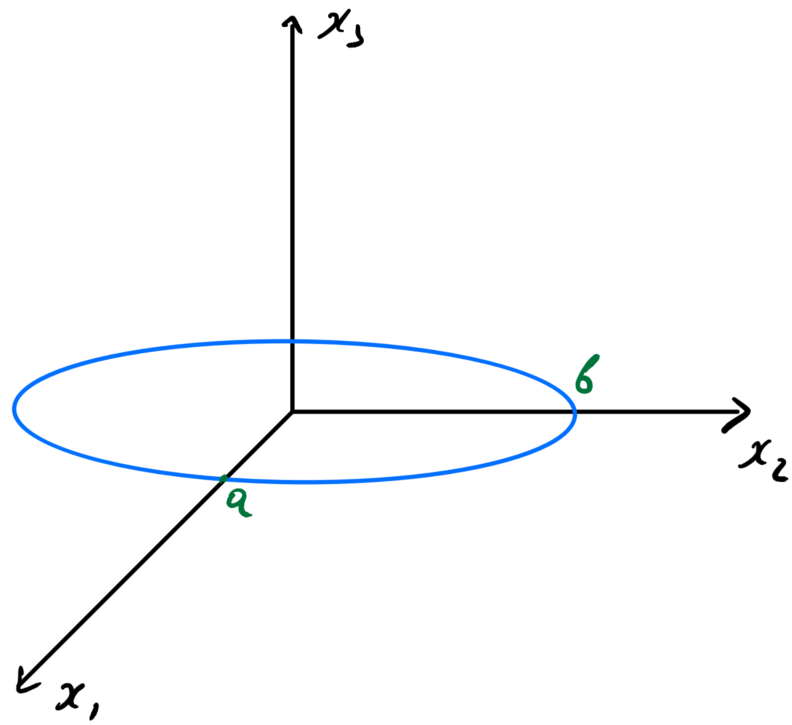
a) $$x(t) = (a_1 + b_1 t,\ a_2 + b_2 t,\ a_3 + b_3 t)$$

- line passing through (a_1, a_2, a_3) and parallel to (b_1, b_2, b_3)



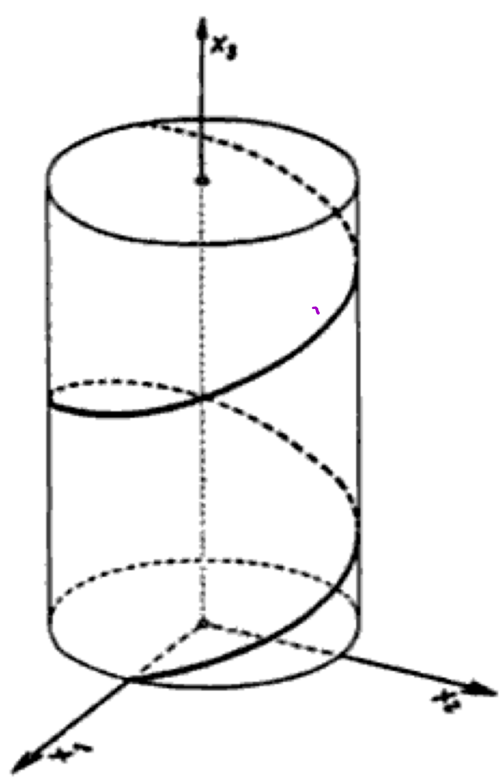
b) $x(t) = (a\cos t, b\sin t, 0)$

- elips



c) $x(t) = (r\cos t, r\sin t, ct)$, $c \neq 0$

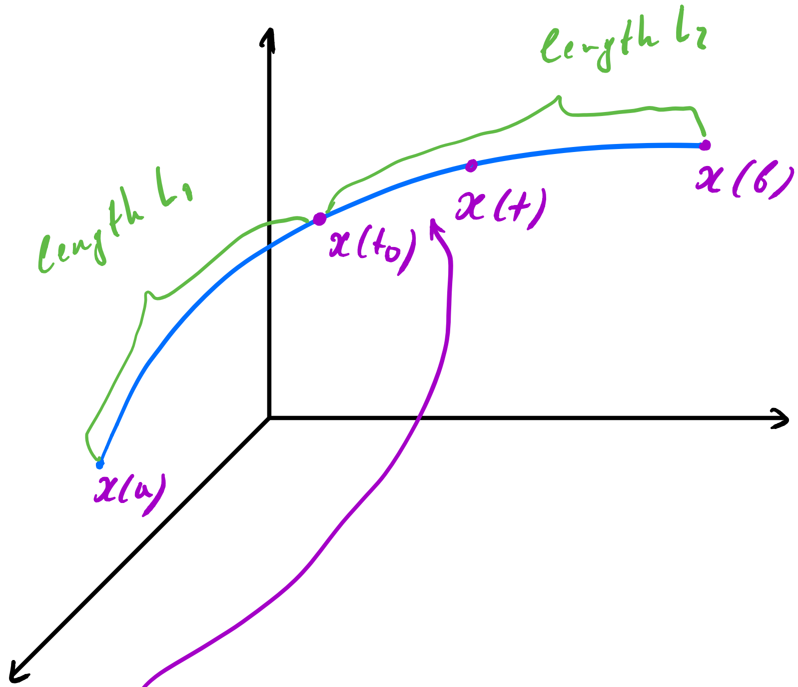
- circular helix.



we recall that

$$L = \int_a^b |x'(t)|\, dt$$

is the length of the curve



$$s(t) = \int_{t_0}^{t} |x'(\tau)|\, d\tau$$

- lenght of the part of the curve

Since $|x'(\tau)| > 0$, the function $s : [a, b] \to [a^*, b^*]$ is strictly increasing, we get that there exists the inverse map

$$t = t(s), \quad s \in [-L_1, L_2]$$

The parametrisation

$$x(s) := x(t(s))$$ is

called a natural parametrization.
Remark that the point to for $s=0$ is chosen arbitrary.

Notations:

$$\dot{x} = \frac{dx}{ds}, \quad \ddot{x} = \frac{d^2x}{ds^2}$$ – for natural parametrization

$$x' = \frac{dx}{dt}, \quad x'' = \frac{d^2x}{dt^2}$$ – for any parametrization.

We remark that

$$\dot{x}(s) = \frac{dx(s)}{ds} = \frac{dx}{dt} \cdot \frac{ds}{dt} =$$

$$= \frac{dx}{dt}(t(s)) \cdot \frac{1}{|x'(t(s))|}.$$

Hence $|\dot{x}(s)| = \frac{1}{|x'(s)|} |x'(s)| = 1.$