22. Spectral representation of bounded self-adjoint operators II.

1. Projections operators

Let H be a Hilbert space, and Y be a closed subspace of H.
In Lecture 18, we have showed that

$$H = Y \oplus Y^{\perp},$$

that is, for every $x \in X$ there exists a unique $y \in Y$ and $z \in Y^{\perp}$ such that

$$x = y + z.$$

We defined y as the minimizer of the function

$$Y \ni \tilde{y} \mapsto \| x - \tilde{y} \|,$$

i.e.

$$\| x - y \| = \inf_{\tilde{y} \in Y} \| x - \tilde{y} \|.$$

We define the operator

$$P : H \to H$$
$$Px := y$$

which is called an orthogonal projection or projection on H. More specifically, P is called the projection of H onto Y.

Exercise 22.1 Show that P is a bounded linear operator on H with $\|P\| = 1$.

Remark 22.2 If P is the projection of H onto Y, then

$$P(H) = \{ Px : x \in H \} = Y$$

and

$$\operatorname{Ker} P = Y^{\perp}.$$

Th 22.3 A bounded linear operator $P: H \to H$ on a Hilbert space H is a projection iff P is self-adjoint and idempotent (that is, $P^2 = P$).

Proof Suppose that P is a projection on H. Set $Y := P(H)$. Then

$$P^2 x = Px.$$

Indeed, $Px \in Y$, and

$$Px = \underset{\overset{\cap}{Y}}{Px} + \underset{\overset{\cap}{Y^{\perp}}}{0}$$

Hence, by the definition of the orthogonal projection

$$P^2x = P(Px) = Px.$$

Let us show that $P^* = P$, that is, $\forall x_1, x_2 \in H$

$$\langle Px_1, x_2 \rangle = \langle x_1, Px_2 \rangle.$$

Take $x_1, x_2 \in H$. Then there exists unique $y_1, y_2 \in Y$, $z_1, z_2 \in Y^\perp$ such that

$$x_1 = y_1 + z_1, \quad x_2 = y_2 + z_2$$

Then

$$\langle Px_1, x_2 \rangle = \langle y_1, y_2 + z_2 \rangle = \langle y_1, y_2 \rangle + \underbrace{\langle \overset{\in Y}{y_1}, \overset{\in Y^\perp}{z_2} \rangle}_{=0}$$

$$\langle x_1, Px_2 \rangle = \langle y_1 + z_1, y_2 \rangle = \langle y_1, y_2 \rangle + \underbrace{\langle \overset{\in Y^\perp}{z_1}, \overset{\in Y}{y_2} \rangle}_{=0}$$

So, $\langle Px_1, x_2 \rangle = \langle x_1, Px_2 \rangle$.

b) Conversely, suppose that $P^2 = P = P^*$. Denote

$$Y := P(H).$$

Then for every $x \in H$

$$x = Px + (x - Px)$$

Let us show that $x - Px \in Y^\perp$.

Take $y \in Y$, then $y = Pv$ for some $v \in H$.

Compute

$$\langle y, x - Px \rangle = \langle Pv, x - Px \rangle =$$

$$= \langle Pv, x \rangle - \langle Pv, Px \rangle =$$

$$= \langle Pv, x \rangle - \langle P^* Pv, x \rangle =$$

$$= \langle Pv, x \rangle - \langle \underset{P}{\underset{\shortparallel}{P^2}} v, x \rangle = 0.$$

Hence, we represented

$$x = \underbrace{Px}_{\in Y} + \underbrace{(x - Px)}_{\in Y^\perp}.$$

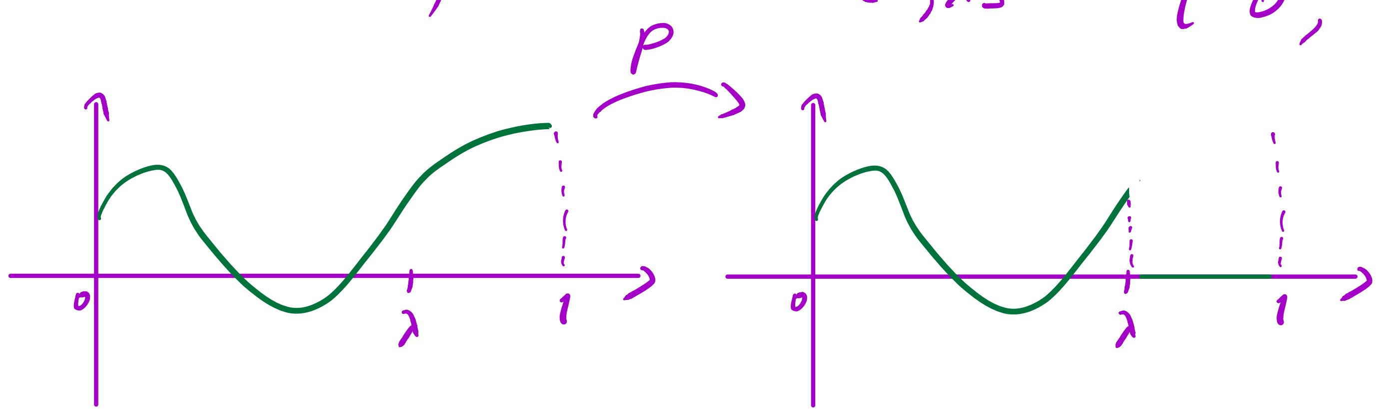
Then, P is the projection onto Y.

$\blacksquare$

Example 22.4 We consider $H = L_2[0,1]$.

Define for $\lambda \in [0,1]$

$$(Px)(t) = x(t)\, \mathbb{1}_{[0,\lambda]}(t) = \begin{cases} x(t), & t \le \lambda \\ 0, & t > \lambda. \end{cases}$$



Then P is a projection on H. Indeed, trivially, $P^2 = P$. Compute

$$\langle Px_1, x_2\rangle = \int_0^1 (Px_1)(t)\,\overline{x_2(t)}\,dt =$$

$$= \int_0^1 x_1(t)\,\mathbb{I}_{[0,\lambda]}(t)\,\overline{x_2(t)}\,dt =$$

$$= \int_0^1 x_1(t)\,\overline{x_2(t)\,\mathbb{I}_{[0,\lambda]}(t)}\,dt =$$

$$= \int_0^1 x_1(t)\,\overline{(Px_2)(t)}\,dt = \langle x_1, Px_2\rangle.$$

Hence, $P^* = P$. Consequently, P is the projection of H onto

$$P(H) = \{ y \in L_2[0,1] : \ y(t) = 0, \ t > \lambda \}.$$

$\blacksquare$

2. Properties of projection operators

Let P_1, P_2, P be projectoons operators on H. Let $Y_i = P_i(H)$, $Y = P(H)$.

1) P is positive and $\langle Px, x\rangle = \|Px\|^2$.

2) $P_1 P_2$ is projection iff $P_1P_2 = P_2 P_1$

Then $P_1 P_2$ projects H onto $Y_1 \cap Y_2$.

3) $P_1 + P_2$ is a projection on H iff $Y_1 \perp Y_2$. In this case $P_1 + P_2$ is the projection of H onto $Y_1 \oplus Y_2$.

4) $P_2 - P_1$ is a projection on H iff $Y_1 \subset Y_2$.

Th. 22.5 (Partial order) Let P_1 and P_2 be projections defined on a Hilbert space H. Denote $Y_i = P_i(H)$. The following conditions are equivalent.

(1) $P_2 P_1 = P_1 P_2 = P_1$

(2) $Y_1 \subset Y_2$

(3) $\operatorname{Ker} P_1 \supset \operatorname{Ker} P_2$

(4) $\| P_1 x \| \leq \| P_2 x \|$

(5) $P_1 \leq P_2$.

3 Spectral family

Let H be a complex Hilbert space.

Def 22.6. • A real *spectral family* is a family $\{E_\lambda, \lambda \in \mathbb{R}\}$ of projections E_λ on H such that

1) $E_\lambda \leq E_\mu$, hence $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$, for $\lambda < \mu$

2) $\lim_{\lambda \to -\infty} E_\lambda x = 0$, $\lim_{\lambda \to +\infty} E_\lambda x = x \quad \forall x \in H.$

3) $E_{\lambda+0} x := \lim_{\mu \to \lambda+0} E_\mu x = E_\lambda x \quad \forall x \in H$

• $\{E_\lambda, \lambda \in \mathbb{R}\}$ is called a *spectral family on an interval* $[a,b]$ if

$$E_\lambda = 0 \quad \text{for } \lambda < a \quad \text{und}$$

$$E_\lambda = I \quad \text{for } \lambda \geq b.$$

We now define a spectral family for a bounded self-adjoint operator $T: H \to H$. We define

$$T_\lambda := T - \lambda I$$

Let $B_\lambda := (T_\lambda^2)^{\frac{1}{2}}$.

The operator

$$T_\lambda^+ = \frac{1}{2}(B_\lambda + T_\lambda)$$

is called the positive part of T_λ.

We define E_λ – projection of H onto $\operatorname{Ker} T_\lambda^+$, $\lambda \in \mathbb{R}$.

Example 22.7 Let $H = L_2[0,1]$

$$(Tx)(t) = t\,x(t).$$

We want to constract E_λ.

Compute

$$(T_\lambda x)(t) = (Tx)(t) - \lambda x(t) =$$

$$= (t-\lambda)\,x(t), \quad t \in [0,T].$$

Then $(T_\lambda^2 x)(t) = (t-\lambda)^2 x(t)$

$$(B_\lambda x)(t) = \sqrt{(t-\lambda)^2}\,x(t) =$$

$$= |t-\lambda|\,x(t), \quad t \in [0,T].$$

Because B_λ is positive and $B_\lambda = T_\lambda^2$.

So, the positive part of T_λ^+ is defined as follows

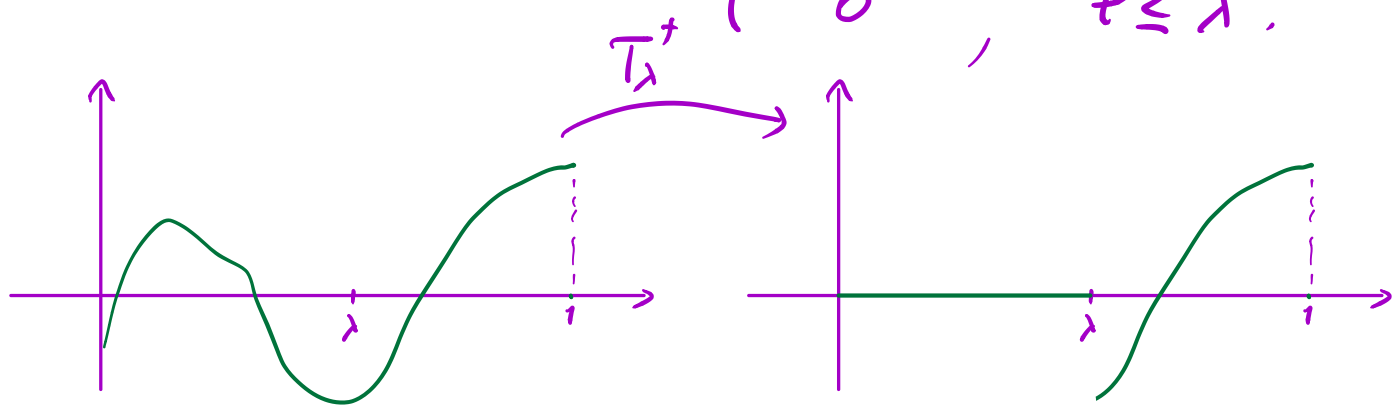
$$(T_\lambda^+ x)(t) = \frac{1}{2}\left[(B_\lambda x)(t) + (T_\lambda x)(t)\right] =$$

$$= \frac{1}{2}\left(|t-\lambda|\, x(t) + (t-\lambda)\, x(t)\right) =$$

$$= (t-\lambda)^+ x(t), \quad t \in [0,1],$$

where $s^+ = \begin{cases} s, & s \geq 0 \\ 0, & s < 0 \end{cases}$.

So, $(T_\lambda^+ x)(t) = \begin{cases} x(t), & t > \lambda \\ 0, & t \leq \lambda. \end{cases}$



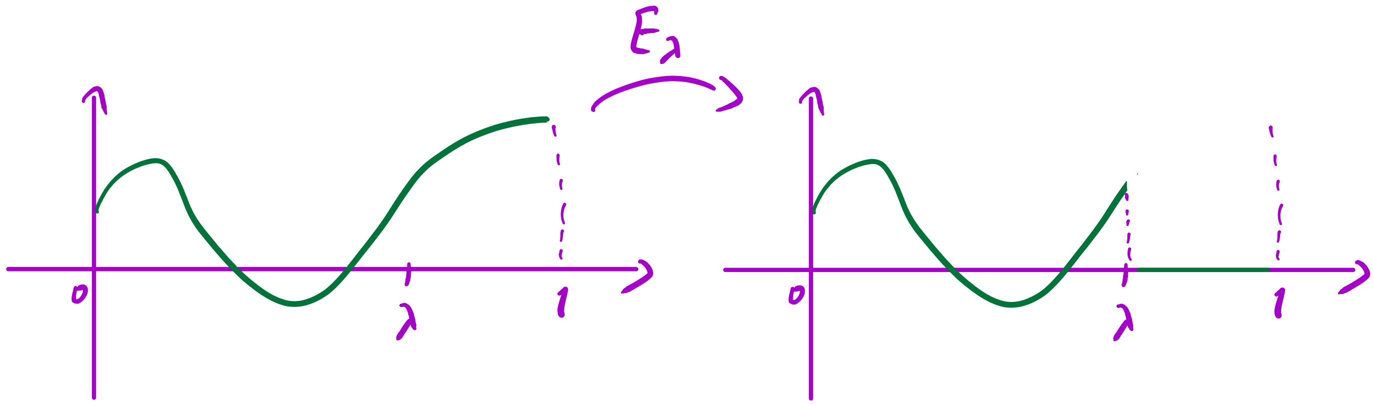
Then $\operatorname{Ker} T_\lambda^+ = \{x : T_\lambda^+ x = 0\} =$

$$= \{x : x(t) = 0, \ t > \lambda\}$$

From Example 22.4. we know that

the projection E_λ of H onto $\operatorname{Ker} T_\lambda^+$ is defined as follows

$$(E_\lambda x)(t) = \mathbb{I}_{[0,\lambda]}(t)\, x(t)$$



Th 22.8. The family $\{E_\lambda, \lambda \in \mathbb{R}\}$, where E_λ is the projection of H onto $\operatorname{Ker} T_\lambda^+$ is the spectral family on the interval $[m, M]$, where m, M is taken from Th. 21.4.

The family $\{E_\lambda, \lambda \in \mathbb{R}\}$ from Th 22.8 is called the spectral family associated with the operator T.

Th 22.9. (Spectral theorem for bounded self-adjoint linear operators)

Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then

$$T = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda = \int_{m-0}^{M} \lambda \, dE_\lambda ,$$

where $\{E_\lambda, \lambda \in \mathbb{R}\}$ is the spectral family associated with T. In particular, $\forall x, y \in H$

$$\langle Tx, y \rangle = \int_{-\infty}^{+\infty} \lambda \, d\langle E_\lambda x, y \rangle =$$

$$= \int_{m-0}^{M} \lambda \, d\langle E_\lambda x, y \rangle.$$

Riemann-Stieltjes integral

Let us come back to the operator

$$(Tx)(t) = t\,x(t).$$

According to Th. 22.9

$$(Tx)(t) = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda x(t) =$$

$$= \int_0^1 \lambda \, d\, \mathbb{I}_{[0,\lambda]}(t) x(t) =$$

$$= t \, x(t).$$

t - fixed