21. Spectral representation of bounded self-adjoint operators

During this lecture we will assume that H is a complex Hilbert space (scalar field $\mathbb{K} = \mathbb{C}$), and $T : H \to H$ is a bounded linear operator.

We recal that T is self-adjoint operator if $T^* = T$, that is,

$$\forall x, y \in H \quad \langle Tx, y \rangle = \langle x, Ty \rangle.$$

1. Spectral representation of self-adjoint operators in finite dimension.

In this section, we will assume that H is finite dimensional. From Math 2 Lecture 12 we know that there exist an orthonormal basis $\{e_1, \dots, e_n\}$ consisting of eigenvalues of T. In particular

$$Te_k = \lambda_k e_k \qquad \forall k = 1, \dots, n$$

and λ_k are called eigenvalues and are real. Since

$$x = \sum_{k=1}^{n} \langle x, e_k \rangle e_k,$$

we get

$$Tx = T\left(\sum_{k=1}^{n} \langle x, e_k \rangle e_k\right) =$$

$$= \sum_{k=1}^{n} \langle x, e_k \rangle Te_k = \sum_{k=1}^{n} \lambda_k \langle x, e_k \rangle e_k.$$

So,

$$Tx = \sum_{k=1}^{n} \lambda_k \langle x, e_k \rangle e_k.$$

Let us define operators

$$P_k x = \langle x, e_k \rangle e_k$$

– projection onto span $\{e_k\}$.

Then

$$Tx = \sum_{k=1}^{n} \lambda_k P_k x, \quad \text{or}$$

$$T = \sum_{k=1}^{n} \lambda_k P_k \qquad (21.1)$$

But this formula can not be extendet to infinite-dimensional Hilbert space. For instance, if

$$H = L^2[0,1], \quad (Tx)(t) = t\,x(t),$$

then $T^* = T$ and $\sigma(T) = \sigma_c(T) = [0,T]$

So, we need to rewrite (21.1) in more appropriate fashion.

Let us assume for simplisity that $\lambda_1 < \lambda_2 < \dots < \lambda_n$.

We introduce

$$E_\lambda = \sum_{k:\,\lambda_k \le \lambda} P_k .$$

Remark, that

$$E_\lambda = 0, \quad \lambda < \lambda_1$$

$$E_\lambda = P_1, \quad \lambda_1 \le \lambda < \lambda_2$$

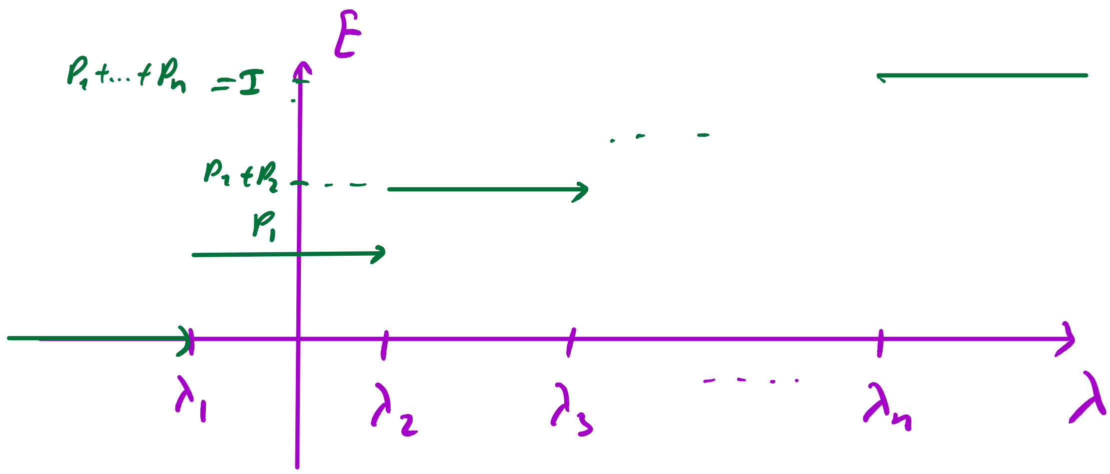
$$E_\lambda = P_1 + P_2, \quad \lambda_2 \le \lambda < \lambda_3$$

$$\dots$$

$$E_\lambda = P_1 + \dots + P_k, \quad \lambda_k \le \lambda < \lambda_{k+1}$$

$$E_\lambda = I \quad \lambda \ge \lambda_n$$

Remark that $E_\lambda x = \sum_{j=1}^{k} \langle x, e_j \rangle e_j$, $\lambda_k \le \lambda < \lambda_{k+1}$, is the projection onto span$\{e_1, \dots, e_k\}$. Moreover it „increases“ and is right continuous.



Then, in particular, $E_{\lambda_k} = P_1 + \dots + P_k$

and $P_k = E_{\lambda_k} - E_{\lambda_k -}$.

Consequently,

$$T = \sum_{k=1}^{n} \lambda_k \left(E_{\lambda_k} - E_{\lambda_k -} \right) =$$

$$= \int_{-\infty}^{+\infty} \lambda \, d E_\lambda .$$ ← Riemann-Stieltjes integral.

We need to understand the last equality as follows

$$\langle Tx, y \rangle = \sum_{k=1}^{n} \lambda_k \left(\langle E_{\lambda_k} x, y \rangle - \langle E_{\lambda_k -} x, y \rangle \right) =$$

$$= \int_{-\infty}^{+\infty} \lambda \, d \langle E_\lambda x, y \rangle .$$

Later we will extend this formula to infinite-dimensional spaces. Namely, we will show that there exists an „increasing" right-continuous family of projection operators $E_\lambda, \lambda \in \mathbb{R}$, such that $E_{-\infty} = 0$, $E_{+\infty} = I$ and

$$T = \int_{-\infty}^{+\infty} \lambda \, d E_\lambda .$$

2. Spectral properties of bounded self-adjoint operators.

Example 21.1 Let $H = L^2[0,1]$ be the Hilbert space of 2-integrable complex-valued functions on $[0,1]$. Consider

$$(Tx)(t) = t\, x(t), \quad t \in [0,1], \quad x \in L^2[0,1]$$

1) T is self-adjoint. Indeed,

$$\langle Tx, y \rangle = \int_0^1 t x(t) \overline{y(t)} \, dt =$$

$$= \int_0^1 x(t) \overline{t y(t)} \, dt = \langle x, Ty \rangle .$$

2) we find the spectrum and resolvent sets. Consider $T_\lambda := T - \lambda I$.

$(T_\lambda x)(t) = (Tx - \lambda x)(t) = t\,x(t) - \lambda\, x(t) =$

$= (t-\lambda)\, x(t) = y(t)$

Then $(R_\lambda y)(t) = \frac{1}{t-\lambda}\, y(t)$, $t \in [0,1]$.

- If $\lambda \in \mathbb{C} \setminus [0,1]$, then $\frac{1}{t-\lambda}$ is a bounded function, so

$$\|R_\lambda y\|^2 = \int_0^1 \frac{1}{|t-\lambda|^2}\, |y(t)|^2\, dt \le$$

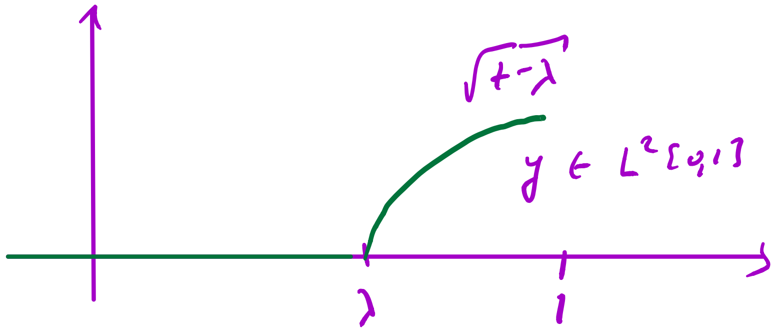
$$\le \sup_{t \in [0,1]} \frac{1}{|t-\lambda|^2} \int_0^1 |y(t)|^2 dt \le$$

$$\le \sup_{t \in [0,1]} \frac{1}{|t-\lambda|^2}\, \|y\|^2 .$$

So, $\|R_\lambda\| \le \sup_{t \in [0,1]} \frac{1}{|t-\lambda|^2}$. Hence R_λ is a bounded linear oper. defined on whole $L^2[0,1] \Rightarrow \lambda \in \rho(T)$.

- If $\lambda \in [0,1]$, then $\frac{1}{t-\lambda}$, $t \in [0,1]$, is not bounded (for $t = \lambda$): and R_λ is not defined on whole $L^2[0,1]$. Say, for

$y(t) = \sqrt{t-\lambda}\, \mathbb{1}_{[\lambda,1]}(t), \quad t \in [0,1]$, we get :



$$R_\lambda y(t) = \frac{\sqrt{t-\lambda}}{t-\lambda} \mathbb{1}_{[\lambda,1]}(t) = \frac{1}{\sqrt{t-\lambda}} \mathbb{1}_{[\lambda,1]}(t).$$

and

$$\| R_\lambda y \|^2 = \int_0^1 \frac{1}{\sqrt{t-\lambda}^{\,2}} \mathbb{1}_{[\lambda,1]}(t)\, dt =$$

$$= \int_\lambda^1 \frac{1}{t-\lambda}\, dt = +\infty \quad \text{if } \lambda < 1.$$

So, R_λ is only defined on the following set:

$$D(R_\lambda) = \left\{ y \in L^2[0,1] : \int_0^1 \frac{|y(t)|^2}{|t-\lambda|}\, dt < +\infty \right\}$$

one can show that $D(R_\lambda)$ is dence in $L^2[0,1]$.

Hence $\lambda \in \sigma_c(T)$. So,

$$\sigma_p(T) = \emptyset, \quad \sigma_c(T) = [0,1], \quad \sigma_r(T) = \emptyset$$

and $\rho(T) = \mathbb{C} \setminus [0,1]$.

Th 21.2 Let H be a complex Hilbert space, $T: H \to H$ be a bounded self-adjoint operator. Then

a) All eigenvalues of T (if they exists) are real

b) Eigenvectors corresponding to different eigenvalues of T are orthogonal.

Th. 21.3. (Resolvent set)
Let H be a complex Hilbert space, and $T: H \to H$ be a bounded self-adjoint linear operator. Then $\lambda \in \rho(T)$ iff there exists $C > 0$ s.t.

$$\|Tx - \lambda x\| \geq C\|x\| \quad \forall x \in H.$$

Th 21.4. (Spectrum) Let H be a complex Hilbert space, and $T: H \to H$ be a bounded self-adjoint linear operator. Then the spectrum $\sigma(T)$ of T is real and belongs to the interval $[m, M]$, where

$$m = \inf_{\|x\|=1} \langle Tx, x \rangle$$

$$M = \sup_{\|x\|=1} \langle Tx, x \rangle .$$

Moreover, m and M are spectral values of T.

Th 21.5. (Residual spectrum) The residual spectrum $\sigma_r(T)$ of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space H is empty.

3. Positive operators

We introduce a partial order „$\leq$" on the set of self-adjoint operators on H. If, T is a self-adjoint, then we know that $\langle Tx, x \rangle$ is real.

Def 21.6. • Let $T_1, T_2 : H \to H$ be bounded self-adjoint operators. We will write $T_1 \leq T_2$ if

$$\langle T_1 x, x \rangle \leq \langle T_2 x, x \rangle \quad \forall x \in H.$$

• A bounded self-adjoint operator T is called *positive*, if

$$T \geq 0,$$

that is equivalent to

$$\langle Tx, x \rangle \geq 0 \quad \forall\, x \in H.$$

We remark that a sum of positive operators is positive.

Th. 21.7 Every positive bounded self-adjoint operator $T : H \to H$ on a complex Hilbert space H has a positive square root $T^{\frac{1}{2}}$, that is,

$$\left(T^{\frac{1}{2}}\right)^2 = T,$$

which is unique. This operator $T^{\frac{1}{2}}$ comutes with every bounded linear operator on H which comutes with T.