Spectral theory of bounded linear operators

We will assume that all spaces are complex ($K = \mathbb{C}$).

1. Basic consepts.

Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subset X$.

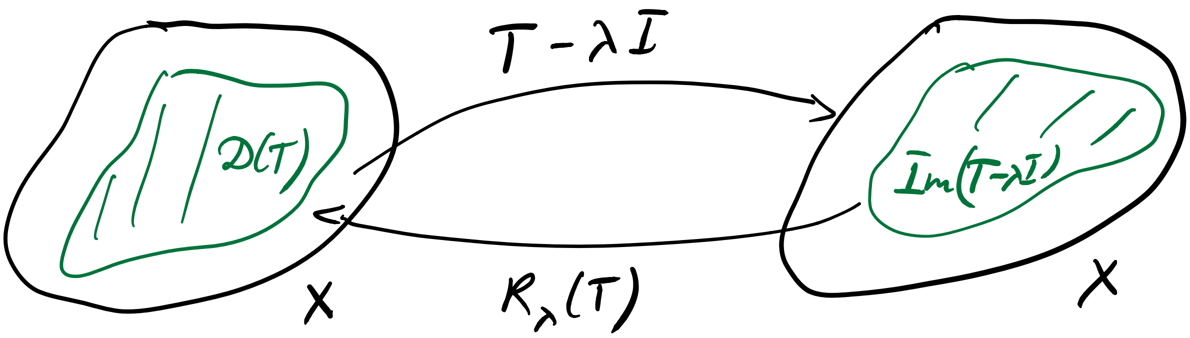
We consider for $\lambda \in \mathbb{C}$ the operator

$$T - \lambda I : D(T) \to X.$$

If it has an inverse, we denote it

$$R_\lambda := R_\lambda(T) = (T - \lambda I)^{-1}$$

Note that R_λ is a linear operator.



Def 20.1 • A regular value of T is a complex number such that

(R1) $R_\lambda(T)$ exists

(R2) $R_\lambda(T)$ is bounded,

(R3) $R_\lambda(T)$ is defined on dence subset of X.

• The resolvent set $\rho(T)$ is the set of all regular values of T.

• The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the spectrum of T.

• $\lambda \in \sigma(T)$ is called a spectral value of T.

The spectrum $\sigma(T)$ is partitioned into three disjoint sets.

Def 20.2 • The point spectrum or discrete spectrum $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exists, that is, (R1) is not satified.

- The continuous spectrum $\sigma_c(T)$ is the set such that $R_\lambda(T)$ exists and (R3) is satisfied but $R_\lambda(T)$ is unbounded.

- The residual spectrum $\sigma_r(T)$ is the set such that $R_\lambda(T)$ exists and the domain of $R_\lambda(T)$ is not dence in X.

Remark that $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$.

We also note that $R_\lambda(T)$ does not exists iff $T - \lambda I$ is not injective. For linear operators it meas that $\exists\, x \neq 0$ s.t.

$$(T-\lambda I)x = Tx - \lambda x = 0.$$

Then $\lambda \in \sigma_p(T)$ iff $\exists\, x \neq 0$ such that

$$Tx - \lambda x = 0.$$

The vector x is called an eigenvector of T (or eigenfunction of T if X is a functional space)

Recall that in a finite-dimensional space X, there exists only point spectrum, $\sigma_c(T) = \sigma_r(T) = \emptyset$.

Example 20.3. Let

$$X := \ell^2 = \left\{ x = (\xi_k)_{k\geq 1} : \xi_k \in \mathbb{C}, \sum_{k=1}^{\infty} |\xi_k|^2 < +\infty \right\}$$

Define

$$T : \ell^2 \to \ell^2$$

$$Tx = (0, \xi_1, \xi_2, \dots), \quad x = (\xi_k)_{k\geq 1}.$$

– right-shift operator, and $\|T\| = 1$ because

$$\|Tx\|^2 = \sum_{k=1}^{\infty} |\xi_k|^2 = \|x\|^2.$$

The operator $R_0(T) = T^{-1} : \operatorname{Im} T \to X$ exists

$$T^{-1} y = (\eta_2, \eta_3, \eta_4, \dots), \quad y = (0, \eta_2, \eta_3, \dots)$$

(It is easy to see that for $\lambda = 0$

$$Tx - 0I = Tx = 0 \Rightarrow x = 0.)$$

But (R3) is not satisfied since

$$\mathcal{D}(R_0(T)) = \operatorname{Im} T = \{ y = (\eta_k)_{k\geq 1} : \eta_1 = 0 \}$$

Here $\lambda = 0$ is residual spectrum.

Prop 20.4. Let X be a complex Banach space, $T \in B(X, X)$ and $\lambda \in \rho(T)$.
Then $R_\lambda(T)$ is defined on the whole X and is bounded.

2. Spectral properties of bounded linear operators

Th. 20.5. Let $T \in B(X, X)$, X be a Banach space. If $\|T\| < 1$, then $(I - T)^{-1}$ exists, belongs to $B(X, X)$ and

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k = I + T + T^2 + \dots,$$

where the series is convergent in $B(X,X)$.

Proof Note that $\|T^k\| \le \|T\|^k$. Since $\|T\| < 1$, then the series

$$\sum_{k=0}^{\infty} \|T^k\| \le \sum_{k=0}^{\infty} \|T\|^k < +\infty .$$

So, the series $\sum_{k=0}^{\infty} \|T^k\|$ is absolutely convergent

Consequently the series

$$S := \sum_{k=0}^{\infty} T^k$$

converges. We compute

$$(I-T)(I+T+T^2+\dots+T^n)=$$
$$=(I+T+T^2+\dots+T^n)(I-T)=$$
$$=I-T^{n+1}.$$

Since $\|T^{n+1}\|\le\|T\|^{n+1}\to 0$ $(\|T\|<1)$ we get

$$(I-T)S=S(I-T)=I.$$

So, $S=(I-T)^{-1}$.

Th 20.6. The resolvent set $\rho(T)$ of $T\in B(X,X)$ on a complex Banach space X is open. Hence the spectrum $\sigma(T)$ is closed.

Th 20.7. The spectrum $\sigma(T)$ of $T\in B(X,X)$ on a complex Banach space X is compact and lies in the disk

$$|\lambda|\le\|T\|. \tag{20.1}$$

Proof Let $\lambda\neq 0$ and $\theta:=\frac{1}{\lambda}$. From Th 20.5 we obtain that

$$R_\lambda = (T-\lambda I)^{-1} = -\frac{1}{\lambda}(I - \theta T)^{-1} =$$

$$= -\frac{1}{\lambda}\sum_{k=0}^{\infty}(\theta T)^k = -\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{1}{\lambda}T\right)^k,$$

where the series converges because

$$\left\|\frac{1}{\lambda}T\right\| = \frac{\|T\|}{|\lambda|} < 1.$$

So, by Th 20.5 $R_\lambda \in B(X,X)$.

Since $\sigma(T)$ is closed, by Th 20.6, and bounded we hew that $\sigma(T)$ is compact ∎

Th 20.8 (Representation theorem for resolvent)

Let X be a Banach space and $T \in B(X,X)$. Then for every $\lambda_0 \in \rho(T)$ the resolvent $R_\lambda(T)$ has the representation

$$R_\lambda = \sum_{k=0}^{\infty}(\lambda-\lambda_0)^k R_{\lambda_0}^{k+1},$$

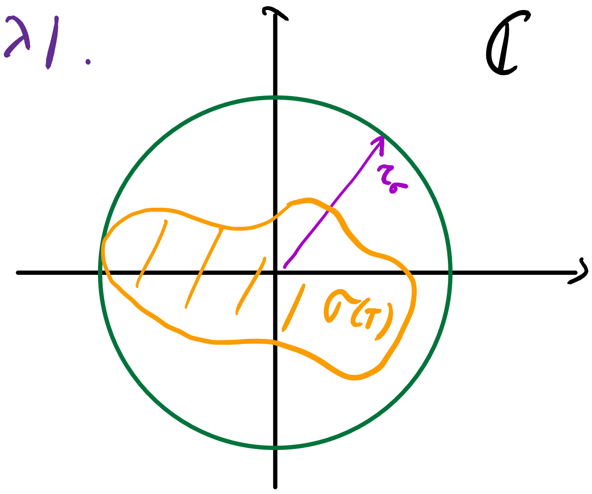
where the series absolutely convergens for λ in the open disk

$$|\lambda-\lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$$

in the complex plain.

Def 20.9. The spectral radius $r_\sigma(T)$ of $T \in B(X,X)$ is the radius

$$r_\sigma(T) = \sup_{\lambda \in \sigma(T)} |\lambda| .$$



From (20.1) it is obvious that

$$r_\sigma(T) \le \|T\|.$$

But one can show that

$$r_\sigma(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|} .$$

Th 20.10 (Resolvent equation, commutativity)
Let X be a complex Banach space, $T \in B(X,X)$, and $\lambda, \mu \in \rho(T)$. Then

a) $R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda$

b) R_λ commutes with any $S \in B(X,X)$ which comutes with T.

c) $R_\lambda R_\mu = R_\mu R_\lambda$.

Proof a) $R_\mu - R_\lambda = R_\mu\big((T - \lambda I) R_\lambda\big) -$

$$-\left(R_\mu (T-\mu I)\right) R_\lambda =$$

$$= R_\mu (T-\lambda I - T + \mu I) R_\lambda = R_\mu (\mu-\lambda) R_\lambda.$$

$$= (\mu-\lambda) R_\mu R_\lambda.$$

b) The assumption $TS = ST$ implies $(T-\lambda I)S = S(T-\lambda I)$. Thus,

$$R_\lambda S = R_\lambda S (T-\lambda I) R_\lambda = R_\lambda (T-\lambda I) S R_\lambda =$$

$$= S R_\lambda.$$

c) R_λ comutes with T by b). Hence R_λ comutes with R_μ.

$\blacksquare$

Th 20.11. Let X be a complex Banach space, $T \in B(X,X)$ and

$$p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \dots + \alpha_0, \quad \alpha_n \neq 0.$$

Then

$$\sigma(p(T)) = p(\sigma(T)),$$

where $p(T) = \alpha_n T^n + \alpha_{n-1} T^{n-1} + \dots + \alpha_0 I$

$p(\sigma(T)) = \{ p(\lambda) \in \mathbb{C},\ \lambda \in \sigma(T) \}.$

Th 20.12. Eigenvectors $x_1, \dots, x_n$ corresponding to different eigenvalues $\lambda_1, \dots, \lambda_n$ of a linear operator T on a vector space X are linearly independent.