18. Orthonormal sets

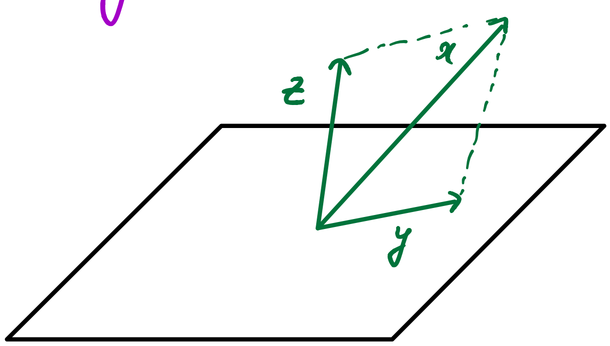
1. Direct sum.

Let X be an inner product space over $\mathbb{K}$. Let Y be a complete subspace of X. Then, we know that the vector

$$z = x - y \perp Y,$$

where

$$\|x-y\| = \inf_{\tilde{y}\in Y} \|x-\tilde{y}\|.$$



Define

$$Y^{\perp} = \{z \in X : z \perp Y\}$$

$Y^{\perp}$ is a subspace of X.

Th 18.1 Let Y be any complete subspace of X. Then for every $x \in X$ there exists unique $y \in Y$ and $z \in Y^{\perp}$ such that

$$x = y + z. \qquad (17.2)$$

Proof The existence of y and z follows from Th. 17.11 and Lemma 17.12. Indeed, take $y \in Y$ s.t.

$$\inf_{\tilde{y}\in Y} \|x-\tilde{y}\| = \|x-y\|$$

and $z = x - y$. Then $z \in Y^{\perp}$ (is orthog. to every vector from Y). So
$$x = y + x - y = y + z.$$
To prove the uniqueness, we assume that
$$x = y + z = y_1 + z_1,$$
where $y, y_1 \in Y$, $z, z_1 \in Z$. Then $\underbrace{y - y_1}_{\in Y} = \underbrace{z_1 - z}_{\in Y^{\perp}}$

So $\langle y - y_1, z_1 - z \rangle = 0$ since $Y \perp Y^{\perp}$.
$$\parallel$$
$$\langle y - y_1, y - y_1 \rangle$$
$\Rightarrow y_1 = y$ by (IP3) and, hence, $z = z_1$.

Def 18.2 A vector space X is said to be the **direct sum** of two subspaces Y and Z of X, written
$$X = Y \oplus Z$$
if every $x \in X$ has a unique representation
$$x = y + z.$$

Remark 18.3 Let Y be a closed subspace. Then $X = Y \oplus Y^{\perp}$.

2. Orthonormal sets

Def 18.4. • An orthogonal set M in X is a subset $M \subset X$ whose elements elements are pairwise orthogonal, that is, $\langle x, y \rangle = 0, \ \forall x, y \in M, \ x \neq y$.

• A set $M \subset X$ is called orthonormal if

$$\langle x, y \rangle = \begin{cases} 1, & x = y \\ 0, & x \neq y. \end{cases} =: \mathbb{1}_{\{x=y\}}$$

Exercise 18.5 a) Show that for every $x, y \in X$, $x \perp y$

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

(Pythagorean relation)

b) Prove that an orthonormal set is linearly independent.

Examples 18.6. a) $X = \mathbb{R}^3$

$$M = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$M = \left\{\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right), \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0\right), (0,0,1)\right\}$$

b) $X = \ell^2$,

$M = \{ e_n, \ n \geq 1 \}$, $e_n = (0, \dots, 0, \underset{\substack{\uparrow \\ n\text{-th position}}}{1}, 0, \dots)$

c) $X = L^2[0, 2\pi]$ - Hilbert space of real-valued functions

$$M = \{ e_n, \ n = 0, 1, 2, \dots \}$$

$$e_0(t) = \frac{1}{\sqrt{2\pi}}, \quad e_n(t) = \frac{\cos nt}{\sqrt{\pi}}$$

$$M = \{ e_n, \ n \geq 1 \}$$

$$e_n(t) = \frac{\sin nt}{\sqrt{\pi}}.$$

Remark 18.7 If $\{e_1, \dots, e_n\} = M$ is a basis in X. Then for every x there exists unique $\alpha_1, \dots, \alpha_n \in K$ such that

$$x = \alpha_1 e_1 + \dots + \alpha_n e_n$$

If M is orthonormal, then α_k, $k = 1, \dots, n$ can be easily founded as follows

$$\langle x, e_k \rangle = \langle \alpha_1 e_1 + \dots + \alpha_k e_k + \dots + \alpha_n e_n, e_k \rangle$$

$$= \alpha_1 \langle e_1, e_k \rangle + \dots + \alpha_k \langle e_k, e_k \rangle + \dots + \alpha_n \langle e_n, e_k \rangle$$

$$= 0 + \dots + \alpha_k + \dots 0 = \alpha_k .$$

So, $\quad x = \langle x, e_1 \rangle e_1 + \dots + \langle x, e_n \rangle e_n .$

We want to extend the idea of the last remark to infinite dimensional inner product spaces.

Let $\{e_1, \dots, e_n\}$ be an orthonormal set. We consider

$$y := \sum_{k=1}^{n} \langle x, e_k \rangle e_k .$$

Let

$$z := x - y .$$

Then

$$x = y + z$$

and $z \perp y$. Indeed,

$$\langle z, y \rangle = \langle x - y, y \rangle = \langle x, y \rangle - \langle y, y \rangle =$$

$$= \sum_{k=1}^{n} \langle x, \langle x, e_k\rangle e_k\rangle - \left\| \sum_{k=1}^{n} \langle x, e_k\rangle e_k \right\|^2 =$$

$$= \sum_{k=1}^{n} \overline{\langle x, e_k\rangle} \langle x, e_k\rangle - \underbrace{\sum_{k=1}^{n} |\langle x, e_k\rangle|^2}_{\text{from Pythagorean relation}} = 0.$$

Hence, again, by Pythagorean relation

$$\|x\|^2 = \|y\|^2 + \|z\|^2 \geq$$

$$\geq \sum_{k=1}^{n} |\langle x, e_k\rangle|^2. \qquad (18.1)$$

(18.1) implies the following statement

Th 18.8 Let $\{e_k, k \geq 1\}$ be an orthonormal sequence in an inner product space X. Then for every $x \in X$

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2$$

(Bessel inequality)

Now we explain how one can obtain an orthonormal sequence if an arbitrary linearly independent sequence is given. Let $\{x_n, n \geq 1\}$ be linearly independent. We construct an orthonormal sequence $\{e_n, n \geq 1\}$ such that $\forall n$

$$\operatorname{span}\{x_1, \dots, x_n\} = \operatorname{span}\{e_1, \dots, e_n\}.$$

Gram - Schmidt procedure:

Set $e_1 := \frac{x_1}{\|x_1\|}$

$$v_2 := x_2 - \langle x_2, e_1 \rangle e_1$$

$$e_2 := \frac{v_2}{\|v_2\|}$$

$$v_3 := x_3 - \langle x_3, e_1 \rangle e_1 - \langle x_3, e_2 \rangle e_2$$

$$e_3 := \frac{v_3}{\|v_3\|}$$

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$$v_n := x_n - \sum_{k=1}^{n-1} \langle x_n, e_k \rangle e_k$$

$$e_n := \frac{v_n}{\|v_n\|}$$

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3. Series related to orthonormal sequences.

Given any orthonormal sequence $\{e_k, k \geq 1\}$ we can consider

$$\sum_{k=1}^{\infty} \alpha_k e_k, \quad \alpha_k \in K. \qquad (18.2)$$

We are going to figure out when the series (18.2) converges.

Th 18.9 Let $\{e_k, k \geq 1\}$ be an orthonormal sequence in a Hilbert space H. Then

(a) Series (18.2) converges iff the series

$$\sum_{k=1}^{\infty} |\alpha_k|^2$$

converges in $\mathbb{R}$

(b) if (18.2) converges and

$$x := \sum_{k=1}^{\infty} \alpha_k e_k,$$

then $\alpha_k = \langle x, e_k \rangle \quad \forall k \geq 1.$

(c) For every $x \in H$, the series

$$\sum_{k=1}^{\infty} \langle x, e_k \rangle e_k \qquad (18.3)$$

converges (but not necessarily to x)

Proof a) Let $S_n = \alpha_1 e_1 + \dots + \alpha_n e_n$ and

$$R_n = |\alpha_1|^2 + \dots + |\alpha_n|^2.$$

Compute for $n < m$

$$\|S_m - S_n\|^2 = \|\alpha_{n+1} e_{n+1} + \dots + \alpha_m e_m\|^2 =$$

$$= |\alpha_{n+1}|^2 + \dots + |\alpha_m|^2 = R_m - R_n.$$

Hence $\{S_n\}_{n\geq 1}$ is a Cauchy sequence in H iff $\{R_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$

So, $\{S_n\}_{n\geq 1}$ converges in H iff $\{R_n\}_{n\geq 1}$ converges in $\mathbb{R}$.

b) Let $x = \sum_{k=1}^{\infty} \alpha_k e_k$. Compute for $k \leq n$

$$\langle S_n, e_k \rangle = \alpha_k$$

Since $S_n \to x$, by the continuity of the inner product

$$\alpha_k = \langle S_n, e_k \rangle \to \langle x, e_k \rangle, \quad n \to \infty.$$

c) From Bessel inequality

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$$

converges. From a) series (18.3) also converges.

$\blacksquare$

4. Total orthonormal sets

Def 18.10 • A set $M \subset X$ is called a **total set** if

$$\overline{\operatorname{span} M} = X,$$

that is, if $\operatorname{span} M$ is dence in X.

• A total orthonormal family in X is called an **orthonormal basis**.

Th 18.11 In every Hilbert space H there exists a total orthonormal set

Th 18.12 An orthonormal set M in a Hilbert space H is total iff $\forall\, x \in H$

$$\sum_k |\langle x, e_k \rangle|^2 = \|x\|^2$$

(Parseval equality)

Th 18.13. Let H be a Hilbert space. Then

a) If H is separable, then every orthonormal set in H is countable

b) If H contains a total orthonormal sequence, then H is separable.