16. Dual spaces.

1. Normed spaces of operators.

Let X, Y be normed spaces.
We consider the set

$$B(X,Y)$$

of all bounded linear operators from X into Y (the domain of operators is X).
$B(X,Y)$ becomes a vector space if we define the sum T_1+T_2, as

$$(T_1+T_2)(x) = T_1 x + T_2 x, \quad x \in X,$$

and multiplication by scalar

$$(\alpha T)(x) = \alpha T x, \quad x \in X,$$

where $T_1, T_2, T \in B(X,Y)$, $\alpha \in K$.

Th. 16.1 The vector space $B(X,Y)$ is a normed space with norm defined by

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|. \qquad (16.1)$$

Exercise 16.2 Prove Th. 16.1, that is, show that the function
$$\|\cdot\| : B(X,Y) \to \mathbb{R}$$
defined by (16.1) is a norm on $B(X,Y)$.

Th. 16.3 If Y is a Banach space, then $B(X,Y)$ is a Banach space.

Proof. Let $T_n \in B(X,Y)$, $n \geq 1$, be a Cauchy sequence in $B(X,Y)$. Then
$$\|T_n - T_m\| \to 0, \quad n,m \to \infty.$$

Take $x \in X$ and consider $T_n x$, $n \geq 1$. Then this sequence is Cauchy in Y. Indeed,
$$\|T_n x - T_m x\| = \|(T_n - T_m)(x)\| \leq$$
$$\leq \|T_n - T_m\| \, \|x\| \to 0, \quad n,m \to \infty$$
Since Y is complete, $\exists y \in Y$ s.t.
$$T_n x \to y, \quad n \to \infty$$
Define $Tx := y$. Let us show that T is linear.

$$T(\alpha x+\beta z)=\lim_{n\to\infty} T_n(\alpha x+\beta z) =$$

$$= \lim_{n\to\infty}\left(\alpha T_n x + \beta T_n z\right) =$$

$$= \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n z =$$

$$= \alpha T x + \beta T z.$$

We next prove that $T \in B(X,Y)$ and $T_n \to T$. Take any $\varepsilon > 0$. Then $\exists\, N$ $\forall\, n,m \geq N$

$$\| T_n - T_m \| < \frac{\varepsilon}{2}$$

Note that for $n \geq N$

$$\|T_n x - Tx\| = \|T_n x - \lim_{m\to\infty} T_m x\| =$$

$$= \lim_{m\to\infty} \|T_n x - T_m x\| \leq \lim_{m\to\infty} \|T_n - T_m\|\|x\|$$

$$\leq \frac{\varepsilon}{2}\|x\| < \varepsilon \|x\|. \qquad (16.2)$$

So, $T_n - T$ is bounded. So,

$T = T_n - (T_n - T)$ is also bounded

Moreover, $\|T_n - T\| < \varepsilon$, by (16.2).

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2. Dual spaces.

Let X be a normed space

Def. 16.4 The set of all bounded linear functionals on X with norm

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)|$$

is called the **dual space** of X and is denoted by X'

Th 16.5 The dual space X' of a normed space X is a Banach space (whether or not X is).

Proof The statement directly follows from Th. 16.3 and the fact that $Y = K$ is a Banach space (with norm $\|\alpha\| = |\alpha|$).

Def 16.6. • An **isomorphism** of a normed space X onto a normed space $\tilde{X}$ is a bijective linear operator $T: X \to \tilde{X}$ which preserves the norm, that is, for all $x \in X$

$$\underbrace{\|Tx\|}_{\text{norm in } \tilde{X}} = \underbrace{\|x\|}_{\text{norm in } X}$$

• if there exists an isomorphism of X onto $\tilde{X}$, then X and $\tilde{X}$ are called *isomorphic normed spaces*.

Examples 16.7 a) $(\ell_n^p)' = \ell_n^q$, where

$$\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p < +\infty.$$

Let $f \in (\ell_n^p)'$. We consider a basis in ℓ_n^p: $e_1 = (1, 0, \dots 0), \dots, e_n = (0, \dots, 0, 1)$.

Then $x = (\xi_1, \dots, \xi_n) = \sum_{k=1}^{n} \xi_k e_k$.

Using the linearity of f we get

$$f(x) = \sum_{k=1}^{n} f(e_k)\, \xi_k = \sum_{k=1}^{n} \gamma_k \xi_k,$$

where $\gamma_k := f(e_k)$, $k = 1, \dots, n$.

Next, we compute the norm of f.

By the Hölder inequality

$$|f(x)| = \left| \sum_{k=1}^{n} \gamma_k \xi_k \right| \le$$

$$\le \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} |\xi_k|^p \right)^{\frac{1}{p}} =$$

$$= \|u\|_q \|x\|_p ,$$

where $u = (\gamma_1, \dots, \gamma_n)$.

Next, take

$$\begin{array}{l} + \text{ if } \gamma_k > 0 \\ - \text{ if } \gamma_k < 0 \end{array}$$

$$x = \left(\pm |\gamma_1|^{q-1}, \dots, \pm |\gamma_n|^{q-1} \right)$$

and compute

$$|f(x)| = \sum_{k=1}^{n} \gamma_k \left(\pm |\gamma_k|^{q-1} \right) =$$

$$= \sum_{k=1}^{n} |\gamma_k|^q$$

$$\|x\| = \left(\sum_{k=1}^{n} |\gamma_k|^{\overbrace{(q-1)p}^{=q}} \right)^{\frac{1}{p}} = \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1-\frac{1}{q}}$$

So, $|f(x)| = \sum_{k=1}^{n} |\gamma_k|^q =$

$$= \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1-\frac{1}{q}} =$$

$$= \|u\| \|x\|$$

Hence, $\|f\| = \|u\|$.

Consequently, the map

$$f \longmapsto (f(e_k))_{k=1}^{n} =: u$$

is an isomorphism of $(\ell_n^p)'$ onto ℓ_n^q

and $\|f\| = \|u\|_q$.

In other words, any bounded linear functional f can be written in the form

$$f(x) = \sum_{k=1}^{n} \gamma_k \xi_k =: \langle u, x \rangle,$$

where $u = (\gamma_k)_{k=1}^{n} \in \ell_n^q$, and $\|f\| = \|u\|_q$.

b) $(\ell_n^1)' = \ell_n^\infty$, $(\ell_n^\infty)' = \ell_n^1$

c) $(\ell^p)' = \ell^q$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

d) $(\ell^1)' = \ell^\infty$

e) $c' = (c_0)' = \ell_1$

f) $(L^p[a,b])' = L^q[a,b]$, $(L^1[a,b])' = L^\infty[a,b]$

g) $(C[a,b])' =$ "functions of bounded variation"

3. Dual space to $C[a,b]$

Def 16.8 A function $w : [a,b] \to \mathbb{R}$ is said to be of **bounded variation** on $[a,b]$ if its total variation

$$\mathrm{Var}(w) = \sup \sum_{j=1}^{n} |w(t_j) - w(t_{j-1})|$$

is finite, where the supremum is taken over all partitions

$$a = t_0 < t_1 < \ldots < t_n = b.$$

Example 16.9 If w is nondecreasing, then w has bounded variation. Indeed,

$$\mathrm{Var}(w) = \sup \sum_{j=1}^{n} |w(t_j) - w(t_{j-1})| =$$

$$= \sup \sum_{j=1}^{n} (w(t_j) - w(t_{j-1})) = w(b) - w(a).$$

Remark 16.10 A function w has bounded variation if it can we written as a difference of two nondecreasing functions, that is, $\exists\, w_1, w_2 : [a,b] \to \mathbb{R}$, nondecreasing such that

$$w = w_1 - w_2 .$$

Let $BV[a,b]$ be the set of all functions on $[a,b]$ of bounded variation.
It is obvious that $BV[a,b]$ is a vector space over $K = \mathbb{R}$.
Define the norm on this space as follows

$$\|w\| = |w(a)| + Var(w)$$

Lemma 16.9 $BV[a,b]$ is a Banach space.

If $x \in C[a,b]$ and $w \in BV[a,b]$, then one can check that there exists the **Riemann - Stieltjes integral**

$$\int_a^b x(t)\, dw(t) = \lim_{\lambda \to 0} \sum_{k=1}^{n} x(\xi_k)\big(w(t_k) - w(t_{k-1})\big),$$

where $\lambda = \max_k |t_k - t_{k-1}|$, $\xi_k \in [t_{k-1}, t_k]$,

$a = t_0 < t_1 < \dots < t_n = b.$

Remark that if $w \in C^1[a,b]$, then $w \in BV[a,b]$ and

$$\int_a^b x(t)\, dw(t) = \int_a^b x(t)\, w'(t)\, dt.$$

Th 16.10 (Functionals on $C[a,b]$) Every $f \in (C[a,b])'$ can be expressed as a Riemann-Stieltjes integral

$$f(x) = \int_a^b x(t)\, dw(t)$$

with $\|f\| = \operatorname{Var}(w)$

If Th 16.11 the function can be made unique if we additionally require that w is right continuous and $w(0)=0$. So,

$$(C[a,b])' = BV_0[a,b],$$

where $BV_0[a,b] \subset BV[a,b]$ contains all righ-continuous functions of bdd. variation.