15. Linear operators

1. Basic definition.

Let X, Y be vector spaces over the same scalar field K

Def. 15.1 A *linear operator* T is a map from $\mathcal{D}(T) \subset X$ to Y such that

1) the domain $\mathcal{D}(T)$ of T is a vector subspace of X;
2) $\forall$ $x, y \in \mathcal{D}(T)$ and scalar α

$$T(x+y) = Tx + Ty$$

$$T(\alpha x) = \alpha Tx.$$

- if $Y = K$, then T is called a *linear functional*.

Examples 15.2 a) $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$. Let

$A = (a_{ij})_{i=1, j=1}^{m, n}$ be an $m \times n$-matrix

Define

$$Tx = Ax, \quad x \in \mathbb{R}^n,$$

that is $Tx = (\eta_1, \dots, \eta_m)$, where

$$\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \quad x = (\xi_1, \dots, \xi_n).$$

Then $D(T) = \mathbb{R}^n$ and T is a linear operator.

b) $X = C[a,b]$, $Y = C[a,b]$

$$(Tx)(t) = \int_a^t x(s)\,ds, \quad t \in [a,b].$$

$D(T) = C[a,b].$

c) $X = C[a,b]$, $Y = C[a,b]$

$$(Tx)(t) = x'(t), \quad t \in [a,b]$$

$D(T) = C^1[a,b] \subset C[a,b].$

d) $X = L^p[a,b]$, $Y = L^p[a,b]$,

$\varphi : [a,b] \to \mathbb{R}$ be a measurable function

$$(Tx)(t) = \varphi(t)\,x(t),$$

$$D(T) = \left\{ x \in L^p[a,b] : \int_a^b |\varphi(t) x(t)|^p dt < +\infty \right\}$$

e) $X = l_\infty$, $Y = \mathbb{R}$,

$$Tx = \lim_{k \to \infty} \xi_k, \quad x = (\xi_k)_{k=1}^{\infty},$$

$D(T) = c$, T is linear functional.

2. Bounded and continuous linear operators

Now we will assume that X, Y are normed spaces over the same scalar field.

Def 15.3 • A linear operator $T : D(T) \to Y$, $D(T) \subset X$, is said to be bounded if there exists a real number $C > 0$ such that

$$\underbrace{\|Tx\|}_{\text{norm in } Y} \le C \underbrace{\|x\|}_{\text{norm in } X} \qquad (15.1)$$

For simplification of notation we use $\|\cdot\|$ for notation of norms in X, Y even if they are different.

• The number

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

is called the norm of T

Exercise 15.4 a) Show that $\|T\|$ is the smallest constant C satisfying (15.1), that is,

$$\|T\| = \min \{ C : \|Tx\| \le C\|x\|, \ \forall x \in D(T) \}$$

b) Show that $\|T\| = \sup\limits_{\substack{x \in \mathcal{D}(T) \\ \|x\|=1}} \|Tx\|$.

Example 15.5. a) $X = Y = C[0,1]$

$$(Tx)(t) = \int_0^t x(s)\,ds, \qquad x \in C[0,1] = \mathcal{D}(T)$$

Show that T is bounded and find its norm.

$$\|Tx\| = \max_{t \in [0,1]} \left| \int_0^t x(s)\,ds \right| \le$$

$$\le \max_{t \in [0,1]} \int_0^t |x(s)|\,ds \le \max_{t \in [0,1]} \int_0^t \|x\|\,ds =$$

$$= \|x\| \max_{[0,1]} t = \|x\|.$$

So, $\|T\| \le 1$. Let us show that $\|T\| = 1$. Take $x = 1$, then $\|x\| = 1$.

Moreover, $(Tx)(t) = \int_0^t 1\,ds = t$.

So, $\|Tx\| = 1 \Rightarrow$

$$\|T\| \geq \frac{\|Tx\|}{\|x\|} = 1.$$

$\Rightarrow \|T\| = 1.$

6) Take $X = Y = C[0,1]$.

$$(Tx)(t) = x'(t), \quad \mathcal{D}(T) = C^1[0,1].$$

Let us show that T is unbounded.

Take $x_n(t) = t^n$, $t \in [0,1]$, $n \geq 1$.

Then $\|x_n(t)\| = \max_{t \in [0,1]} (t^n) = 1$.

Compute

$$(Tx_n)(t) = n\, t^{n-1}.$$

Hence, $\|Tx_n\| = \max_{t \in [0,1]} |n t^{n-1}| = n$.

So, $\|T\| \geq \dfrac{\|Tx_n\|}{\|x_n\|} = n \quad \forall n \geq 1$.

$\Rightarrow \|T\| = +\infty$.

(In other words there is no constant c s.t. $\|Tx_n\| \leq C\|x_n\| \quad \forall n \geq 1$ since $n = \dfrac{\|Tx_n\|}{\|x_n\|} \leq C$)

Th. 15.6. Let X be a finite dimensional normed space and T be a linear operator on X. Then T is bounded.

Let $T: \mathcal{D}(T) \to Y$ be a linear operator. We recall that T is continuous at $x_0 \in \mathcal{D}(T)$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in \mathcal{D}(T) \quad \|x - x_0\| < \delta$$
$$\Rightarrow \|Tx - Tx_0\| < \varepsilon.$$

Th 15.7. Let $T: \mathcal{D}(T) \to Y$ be a linear operator. Then

a) T is continuous if and only if T is bounded

b) if T is continuous at a single point, then it is continuous.

Proof a) For $T = 0$ the statement is trivial. Let $T \neq 0$. Then $\|T\| \neq 0$.

$\Rightarrow$) Assume that T is bounded. Take any $x_0 \in \mathcal{D}(T)$ and $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{\|T\|}$.

Then for all $x \in \mathcal{D}(T)$, $\|x - x_0\| < \delta$

we have

$$\|Tx - Tx_0\| = \|T(x-x_0)\| \leq \|T\| \|x-x_0\| <$$

$$< \|T\| \delta = \|T\| \frac{\varepsilon}{\|T\|} = \varepsilon.$$

Since $x_0 \in \mathcal{D}(T)$ was arbitrary, T is continuous

$\Leftarrow)$ Let T be continuous at any point $x_0 \in \mathcal{D}(T)$. Fix any $x_0 \in \mathcal{D}(T)$ and take $\varepsilon = 1$, then $\exists\, \delta > 0$ s.t.
$\forall\, x \in \mathcal{D}(T)$ $\|x - x_0\| < \delta$ it follows
$\|Tx - Tx_0\| < 1$.

Now take any $y \neq 0$ from $\mathcal{D}(T)$ and set

$$x = x_0 + \frac{\delta}{\|y\|} y. \quad \text{Then}$$

$$x - x_0 = \frac{\delta}{2\|y\|} y.$$

Hence $\|x - x_0\| = \frac{\delta}{2} < \delta$.

Then $1 > \|Tx - Tx_0\| = \|T(x-x_0)\| = \|T(\frac{\delta}{2\|y\|} y)\| =$

$$= \frac{\delta}{2\|y\|} \|Ty\|$$

Thus, $\frac{\delta}{2\|y\|} \|Ty\| < 1 \Rightarrow \|Ty\| < \frac{2}{\delta} \|y\|$.

Since, $y \in \mathcal{D}(T)$ was arbitrary, it implies that T is bounded. Remark that here we used the continuity of T only at one point x_0.

b) Hence, by proof in a) if T is continuous at x_0, then it is bounded. Then T is continuous (on $\mathcal{D}(T)$). ∎

Corollary 15.8. Let T be a bounded linear operator. Then

a) if $x_n \to x$ (where $x_n, x \in \mathcal{D}(T)$) then $Tx_n \to Tx$

b) The null set $\mathrm{Ker}(T) = \{x : Tx = 0\}$ is closed.

Exercise 15.9. Prove Corollary 15.8.

Th. 15.9. Let $T : \mathcal{D}(T) \to Y$ be a bounded linear operator, and Y be a Banach space. Then T has an extension

$$\tilde{T} : \overline{\mathcal{D}(T)} \to Y,$$

where $\tilde{T}$ is is a bounded linear operator and $\|\tilde{T}\| = \|T\|$.

Proof We only show how $\tilde{T}$ can be constructed. Let $x \in \overline{D(T)}$. Then there exists a sequence $x_n \in D(T)$ s.t. $x_n \to x$. Since T is linear and bounded then

$$\|Tx_n - Tx_m\| \leq \|T(x_n - x_m)\| \leq \|T\| \, \|x_n - x_m\| \to 0, \quad n, m \to \infty.$$

So, $\{Tx_n\}_{n \geq 1}$ is a Cauchy sequence in Y. Since Y is a Banach space (complete normed space), there exists $y \in Y$ such that

$$Tx_n \to y, \quad n \to \infty.$$

Set $\tilde{T}x := y$.

Show that $\tilde{T}x$ is well-defined. If z_n, $n \geq 1$, is other sequence from $D(T)$ converging to x. Then

$$Tz_n \to y'.$$

Show that $y = y'$. Consider the sequence v_n: $x_1, z_1, x_2, z_2, x_3, z_3, \ldots$

Then this sequence converges to x.

And $Tv_n \to y''$. But

$$\left.\begin{array}{l} Tv_{2k+1} \to y = y'' \\ Tv_{2k} \to y' = y'' \end{array}\right\} \implies y = y'.$$

∎