14. Normed and Banach spaces

1. Vector spaces

We recall from Math 2, Lecture 2
Let $K = \mathbb{R}$ or $\mathbb{C}$

Def 14.1 A *vector space* over a field of scalars K is a non-empty set X of elements $x, y, \ldots$ (called vectors) together with operations addition „+" and multiplication „·" satisfying the following conditions

1) $x + y = y + x$, $x, y \in X$
2) $(x+y)+z = x+(y+z)$, $x, y, z \in X$
3) $\exists$ vector $0 \in X$ s.t. $\forall x \in X$ $0 + x = x$
4) $\forall x \in X$ $\exists y \in X$ (denoted by $-x$) s.t.
$$x + y = 0$$
5) $1 \cdot x = x$, $x \in X$
6) $\alpha(x+y) = \alpha x + \alpha y$, $(\alpha+\beta)x = \alpha x + \beta x$,
$x, y \in X$, $\alpha, \beta \in K$.

We also recall that $Y \subset X$ is called a vector subspace of X if $\forall x, y \in Y$, $\forall \alpha, \beta \in K$ $\alpha x + \beta y \in Y$.

Examples 14.2 The following sets together with operations „+" and „·" are vector spaces

a) $K^n = \{ (\xi_1, \dots, \xi_n) : \xi_k \in K, k = 1, \dots, n \}$

$$(\xi_1, \dots, \xi_n) + (\eta_1, \dots, \eta_n) = (\xi_1 + \eta_1, \dots, \xi_n + \eta_n)$$

$$\alpha(\xi_1, \dots, \xi_n) = (\alpha\xi_1, \dots, \alpha\xi_n),$$

where $K = \mathbb{R}$ or $\mathbb{C}$.

b) $C[a,b] = \{ x : [a,b] \to \mathbb{R} : x \text{ - continuous} \}$

$$(x+y)(t) = x(t) + y(t)$$

$$(\alpha x)(t) = \alpha x(t).$$

c) $\ell^p = \{ x = (\xi_1, \xi_2, \dots) : \xi_k \in \mathbb{R}, \sum_{k=1}^{\infty} |\xi_k|^p < \infty \}$

$\ell^\infty = \{ x = (\xi_1, \xi_2, \dots) : \xi_k \in \mathbb{R}, \sup_k |\xi_k| < +\infty \}$

$$x + y = (\xi_1 + \eta_1, \xi_2 + \eta_2, \dots)$$

$$\alpha x = (\alpha \xi_1, \alpha \xi_2, \dots)$$

d) $L^p[a,b] = \{ x : [0,1] \to \mathbb{R} : x \text{ - measurable}, \int_0^1 |x|^p dx < +\infty \}$

We identify $x, y \in L^p$ if $x = y$ λ-a.e., where λ is the Lebesgue measure on $[0,1]$.

$$(x+y)(t) = x(t) + y(t)$$

$$(\alpha x)(t) = \alpha x(t).$$

2. Normed and Banach spaces

Def 14.3 • A **norm** on a vector space X is a real-valued function on X whose value at $x \in X$ is denoted by

$$\|x\|$$

and which has the properties

(N1) $\|x\| \geq 0 \qquad \forall x \in X$

(N2) $\|x\| = 0 \iff x = 0$

(N3) $\|\alpha x\| = |\alpha| \, \|x\| \qquad \forall x \in X, \alpha \in K$

(N4) $\|x+y\| \leq \|x\| + \|y\| \qquad \forall x, y \in K$

(Triangle inequality)

• A **normed space** X is a vector space with a norm definite on it.

Let $(X, \|\cdot\|)$ be a normed vector space. The norm $\|\cdot\|$ defines the metric d on X which is given by

$$d(x,y) = \|x-y\|, \quad x,y \in X$$

One can check that d is a metric on X. The metric d is called the metric induced by the norm.

We will also consider every normed space $(X, \|\cdot\|)$ as a metric space with the metric induced by the norm. So, $\{x_n\}_{n\geq 1}$ converges in X if

$$\|x_n - x\| \to 0, \quad n \to \infty.$$

Similarly, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if

$$\|x_n - x_m\| \to 0, \quad n,m \to \infty.$$

Def 14.4. A normed space $(X, \|\cdot\|)$ is called a Banach space if it is complete in metric induced by the norm $\|\cdot\|$.

Exersice 14.5 Show that a norm satisfies the inequality

$$\big| \|x\| - \|y\| \big| \le \|x-y\|. \quad (14.1)$$

Inequality (14.1) implies that the map

$$X \ni x \longmapsto \|x\| \in \mathbb{R}$$

is continuous.

Examples 14.6.

a) Euclidian space $\mathbb{R}^n$ and unitary space $\mathbb{C}^n$.

$$\|x\| = \left(\sum_{k=1}^{n} |\xi_k|^2 \right)^{\frac{1}{2}}.$$

b) Sequence spaces ℓ^∞, ℓ^p

Norm in ℓ^∞: $\|x\| = \sup_{k\ge 1} |\xi_k|$

Norm in ℓ^p: $\|x\| = \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{\frac{1}{p}}$

c) Space c

$$\|x\| = \sup_{k\ge 1} |\xi_k|$$

Remark that c is a subspace of ℓ_∞.

d) Space $B(A)$

$$\|x\| = \sup_{t \in A} |x(t)|$$

e) Space $C[a,b]$

$$\|x\| = \max_{t \in [a,b]} |x(t)|$$

f) Spaces ℓ_n^p, $p \geq 1$; ℓ_n^∞

ℓ_n^p : $\|x\| = \left(\sum_{k=1}^{n} |\xi_k|^p \right)^{\frac{1}{p}}$; ℓ_n^∞ : $\|x\| = \max_{k=1,\dots,n} |\xi_k|$

g) Spaces $L_p[a,b]$, $p \geq 1$.

L_p : $\|x\| = \left(\int_a^b |x(t)|^p dt \right)^{\frac{1}{p}}$

All spaces in a) - g) are Banach spaces.

Example 14.7 (Incomplete normed space)

The space $C[a,b]$ with norm

$$\|x\| = \int_0^1 |x(t)| dt$$

is incomplete normed space.

3. Finite dimensional normed spaces

Def. 14.8. • Vectors $x_1, \dots, x_n \in X$ are *linearly independent* if the equality

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0$$

only holds if $\alpha_1 = \dots = \alpha_n = 0$.

• A subset $M \subset X$ is *linearly independent* if every nonempty finite subset of M is linearly independent.

• A vector space X is *finite dimensional* if $\exists\, n \geq 1$ such that X contains a linearly independent set of vectors and every set containing more than n vectors is linearly dependent.
The number n is called the *dimension* of X (write $n = \dim X$). If such n does not exists, then X is infinete dimensional.

• If $n = \dim X$, then a linearly indep. family of vectors $\{e_1, \dots, e_n\}$

is called a **basis** for X. If $\{e_1, \dots, e_n\}$ is a basis then for every vector $x \in X$ there exist unique set of scalars $\alpha_1, \dots, \alpha_n$ s.t.
$$x = \sum_{k=1}^{n} \alpha_k e_k.$$

- We say that $Y \subseteq X$ is a **subspace** of a normed space X if Y is a vector subspace of X and the norm on Y is the restriction of the norm on X.

For example c is a subspace of ℓ^∞.

Y is a **closed subspace** of X if additionally Y is a closed subset of X.

4. Schauder basis

In a normed space we can use serieses.

Let $x_n \in X$, $n \geq 1$. We define the partial sums
$$S_n = \sum_{k=1}^{n} x_k = x_1 + \dots + x_n$$

We say that the series $\sum_{n=1}^{\infty} x_n$ **converges** if $\{S_n\}_{n\geq 1}$ is convergent, that is $\exists S \in X$ s.t. $S_n \to S$, $n \to \infty$.

The element S is called the **sum** of the series $\sum_{n=1}^{\infty} x_n$.

A series $\sum_{n=1}^{\infty} x_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$.

Exercise 14.9 Show that the absolute convergence implies the convergence in X if and only if X is a Banach space.

Def. 14.10. If a normed space X contains a sequence $\{e_n\}_{n\geq 1}$ with the property that for every $x \in X$ there exists a unique sequence of scalars $\{\alpha_n\}_{n\geq 1}$ such that

$$x = \sum_{k=1}^{\infty} \alpha_k e_k,$$

then $\{e_n\}_{n\geq 1}$ is called a **Schauder basis** (or basis) for X.

Exercise 14.11. Show that if a normed space has a Schauder basis then X is separable.

The inverse statement in not true in general.

Example 14.12 $\{ e_n = (0, 0, \dots, 0, 1, 0, \dots), n \geq 1 \}$ (1 at the n-th position)

is a Schauder basis for $\ell^p, p \geq 1$.