13. Completeness of metric spaces.

1. Cauchy sequences

We recall the definition of convergence in a metric spaces

Def 12.12 • A sequence $\{x_n\}_{n\geq 1}$ in a metric space $X=(X,d)$ is said to converge or to be convergent if there exists $x\in X$ such that

$$\lim_{n\to\infty} d(x_n, x)=0.$$

• x is called the limit of $\{x_n\}_{n\geq 1}$ and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.

Def 13.1. • A sequence $\{x_n\}_{n\geq 1}$ is said to be a Cauchy sequence if

$$\forall \varepsilon>0 \quad \exists N \quad \forall n,m\geq N \quad d(x_m, x_n)<\varepsilon,$$

in other words if $d(x_n, x_m)\to 0$, $m,n\to\infty$.

• The space is said to be complete if every Cauchy sequence in X converges, that is it has a limit which is an element of X.

Example 13.2 a) All spaces 1) - 9) in Lecture 11 are complete.

b) $X = \mathbb{Q}$, $d(x,y) = |x-y|$, $x,y \in \mathbb{Q}$ is incomplete. Take

$$x_n = \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q}$$

We know that $\sum_{k=0}^{\infty} \frac{1}{k!} = e \notin \mathbb{Q}$.

The sequence, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence indeed, for $n < m$

$$d(x_n, x_m) = |x_n - x_m| = \sum_{k=n+1}^{m} \frac{1}{k!} \leq$$

$$\leq \sum_{k=n+1}^{\infty} \frac{1}{k!} \to 0, \quad n, m \to \infty.$$

But $\{x_n\}_{n\geq 1}$ is not convergent in $X = \mathbb{Q}$ because there exists no $x \in \mathbb{Q}$ such that $x_n \to x$ in $X = \mathbb{Q}$ ($x_n \to e \notin \mathbb{Q}$).

c) $X = (0,1)^2 = \{(\xi_1, \xi_2) : \xi_1, \xi_2 \in (0,1)\}$.

This space is incomplet. Indeed,

Take $x_n = \left(\frac{1}{n}, \frac{1}{n}\right) \in X$, $n \geq 2$.

Then $d(x_n, x_m) = \sqrt{\left(\frac{1}{n} - \frac{1}{m}\right)^2 + \left(\frac{1}{n} - \frac{1}{m}\right)^2} =$

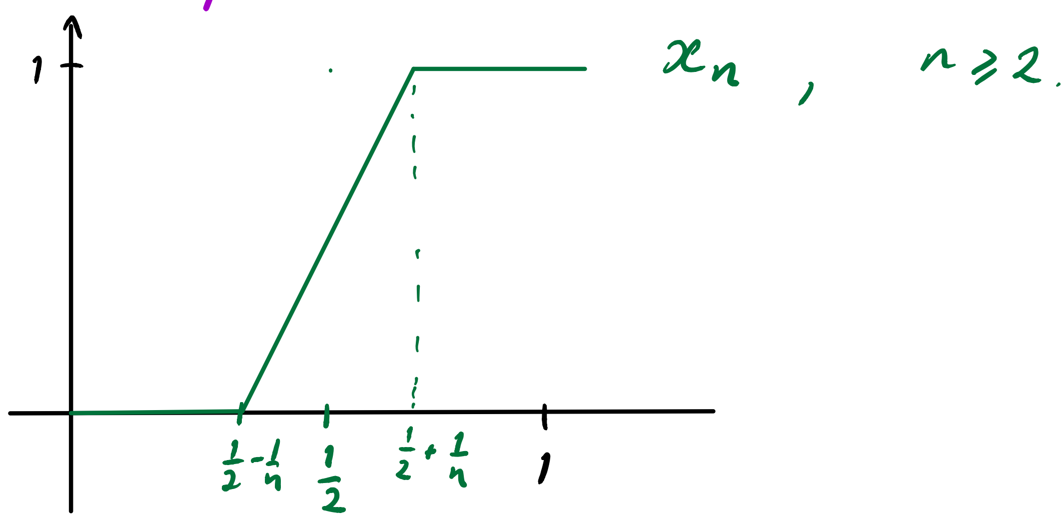
$$= \sqrt{2}\left|\frac{1}{n} - \frac{1}{m}\right| \to 0, \quad n, m \to \infty$$

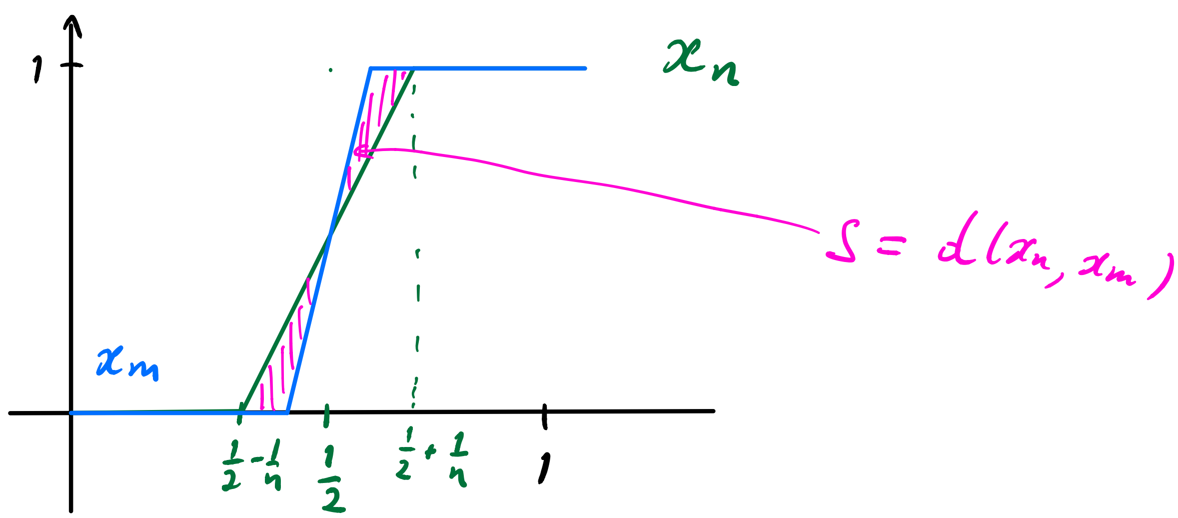
Hence, $\{x_n\}_{n \geq 2}$ is a Cauchy sequence but $\nexists\ x \in X$ s.t. $x_n \to x$ (because $x_n \to (0,0) \notin X$).

d) Let $X = C[0,1]$ and

$$d(x,y) = \int_0^1 |x(t) - y(t)|\, dt$$

Then (X, d) is a metric space (check this!) X is not complete. Take





Hence $\{x_n\}_{n\geq 1}$ is a Cauchy sequence but it does not converges in $C[0,1]$

$$x_n \to \begin{cases} 1, & x \geq \frac{1}{2} \\ 0, & x < \frac{1}{2} \end{cases} \notin C[0,1].$$

2. Some properties.

Th. 13.3 Every convergent sequence in a metric space is a Cauchy sequence.

Proof Let $\{x_n\}_{n\geq 1}$ converges to x.

Then $0 \leq d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \to 0,$
$n, m \to \infty.$

$\blacksquare$

Exercise 13.4 Show that a Cauchy sequence is bounded

Example 13.5. Let us show that ℓ_p is a complete metric space. Take a Cauchy sequence $x_n = (\xi_k^n)_{k=1}^{\infty} \in \ell^p$.

1) Show that $\forall \ell$ $\{\xi_\ell^n\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$. Indeed,

$$|\xi_\ell^n - \xi_\ell^m| = \left(|\xi_\ell^n - \xi_\ell^m|^p\right)^{\frac{1}{p}} \leq$$

$$\leq \left(\sum_{k=1}^{\infty} |\xi_k^n - \xi_k^m|^p\right)^{\frac{1}{p}} = d(x^n, x^m) \to 0,$$
$$m, n \to \infty.$$

So, $\{\xi_\ell^n\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete (see Math1 Th. 5.3), $\exists\ \xi_\ell \in \mathbb{R}$ such that $\xi_\ell^n \to \xi_\ell$, $n \to \infty$.

2) Show that $x = (\xi_k)_{k=1}^{\infty} \in \ell^p$ and

$x_n \to x$ in ℓ^p.

Take $\varepsilon > 0$. By the fact that $\{x_n\}_{n\geq 1}^{\infty}$ is a Cauchy sequeness, we have that

$\exists\ N \geq 1 \quad \forall n, m \geq 1$

$$d(x_n, x_m) < \frac{\varepsilon}{2}.$$

So, $\left(\sum_{k=1}^{\infty} |\xi_k^n - \xi_k^m|^p\right)^{\frac{1}{p}} < \frac{\varepsilon}{2} \quad \Rightarrow$

$$\sum_{k=1}^{\infty} |\xi_k^n - \xi_k^m|^p < \frac{\varepsilon^p}{2^p}$$

By Fatou's lemma

$$\sum_{k=1}^{\infty} \lim_{m\to\infty} |\xi_k^n - \xi_k^m|^p = \sum_{k=1}^{\infty} |\xi_k^n - \xi_k|^p \le \frac{\varepsilon^p}{2^p} < \varepsilon^p.$$

So, $\left(\sum_{k=1}^{\infty} |\xi_k^n - \xi_k|^p\right)^{\frac{1}{p}} < \varepsilon. \quad \forall n \ge N.$

we need only to show that $x = (\xi_k)_{k=1}^{\infty} \in \ell^p$.

By Fatou's lemma

$$\sum_{k=1}^{\infty} |\xi_k|^p = \sum_{k=1}^{\infty} \lim_{n\to\infty} |\xi_k^n|^p \le$$

$$\le \varliminf_{n\to\infty} \sum_{k=1}^{\infty} |\xi_k^n|^p = \varliminf_{n\to\infty} d(0, x_n) < +\infty$$

because $\{x_n\}_{n\ge1}$ is bounded.

Th. 13.6 Let $M \subset X$ be non-empty.

Then

a) $x \in \overline{M}$ if and only if $\exists x_n \in M$, $n \ge 1$, such that $x_n \to x$

b) M is closed if and only if $\forall \{x_n\}_{n\ge1} \subset M$ s.t. $x_n \to x$ in X we have that $x \in M$.

Th. 13.7 Let (X, d) be a complete metric space and $M \subset X$.
The metric subspace (M, d) is complete if and only if M is a closed subset of X.

Proof. $\Rightarrow)$ Given: (M, d) is complete.

We prove that M is closed in X.
We use Th. 13.4 b). Take a subsequence $\{x_n\}_{n\geq 1} \subset M$ such that $x_n \to x$ in X.

Then by Th 13.3 $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X, that is

$$d(x_n, x_m) \to 0 \quad , n, m \to \infty.$$

But then $\{x_n\}_{n\geq 1}$ is Cauchy sequence in (M, d).
Since M is complete $\exists\, y \in M$ such that $x_n \to y$ in M, that is,

$$d(x_n, y) \to 0 \quad , n \to \infty.$$

But then $x_n \to y$ in X. Since the limit can be only unique (see Lemma 12.14), $x = y \in M$.

$\Leftarrow)$ Given : M is closed in X and (X, d) is complete. Take $\{x_n\}_{n\geq 1}$ a Cauchy sequence in M, then $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X. By the completeness of X, $\exists\, x \in X$ such that $x_n \to x$ in X, $n \to \infty$.

But then by Th 6), $x \in M$. So, $x_n \to x$ in M, $n \to \infty$.

$\blacksquare$

Def. 13.8. (Isometric spaces)

a) A map $T : X \to \tilde{X}$ is said to be **isometric** if T preserves distances, that is, if for all $x, y \in X$

$$\tilde{d}(Tx, Ty) = d(x, y).$$

b) The space X is said to be **isometric** with the space $\tilde{X}$ if there exists a bijective isometry of X onto $\tilde{X}$. X, $\tilde{X}$ are called **isometric spaces**

Th 13.9. For a metric space $X = (X, d)$ there exists a complete metric space $\hat{X} = (\hat{X}, \hat{d})$ which has a subspace W

that is isometric with X and dence in $\hat{X}$.
This metric space $\hat{X}$ is unique except for isometries