12. Convergence in metric spaces

1. Continuous maps

Let X be a metric space with metric d, that is, X is a set and d is a function $d : X \times X \to \mathbb{R}$ such that

(M1) $d(x,y) \in [0, +\infty)$

(M2) $d(x,y) = 0 \iff x = y$

(M3) $d(x,y) = d(y,x)$ (symmetry)

(M4) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

Recall that a set G is said to be open if $\forall\, x \in G$ $\exists\, r > 0$ s.t.

$$B_r(x) = \{ y \in X : d(x,y) < r \} \subset G$$

Def 12.1 • Let (X, d_X), (Y, d_Y) be metric spaces. A map

$$T : X \to Y$$

is said to be **continuous at x_0** if $\forall\, \varepsilon > 0$ $\exists\, \delta > 0 : \forall\, x_0 \in X$, $d_X(x, x_0) < \delta$

$$\Rightarrow \quad d_Y(Tx_0, Tx) < \varepsilon.$$

- A function T is continuous on X if T is cont. at every point of X.

Example 12.2. The function $T: \ell^\infty \to \mathbb{R}^2$, defined as

$$Tx = (\xi_1, \xi_2), \quad x = (\xi_k)_{k=1}^\infty;$$

is continuous. Indeed, let $x = (\xi_k)_{k=1}^\infty \in \ell^\infty$, and $\varepsilon > 0$. Then for all $y = (\eta_k)_{k=1}^\infty \in \ell^\infty$

$$d_{\ell^\infty}(x, y) = \sup_k |\xi_k - \eta_k| < \delta,$$

where δ will be chosen later, we have

$$d_{\mathbb{R}^2}(Tx, Ty) = d_{\mathbb{R}^2}((\xi_1, \xi_2), (\eta_1, \eta_2)) =$$

$$= \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2} <$$

$$< \sqrt{\delta^2 + \delta^2} = \sqrt{2\delta^2} = \sqrt{2}\,\delta = \varepsilon$$

Here $\delta = \frac{\varepsilon}{\sqrt{2}}$.

So, T is continuous at all $x \in \ell^\infty$. Hence, T is continuous on ℓ^∞.

Th 12.3. A map $T: X \to Y$ is continuous on X if and only if $\forall G$ - open in Y

$$f^{-1}(G) = \{x \in X : f(x) \in G\}$$

is open in X.

Def 12.4 • A point x_0 is called a **limit point** of a set $M \subset X$ if

$$\forall \varepsilon > 0 \quad \exists x \neq x_0, \; x \in M \text{ s.t. } x \in B_\varepsilon(x_0)$$

• The set $\overline{M}$ which contains all points of M and all limit points is called the **closure of M**.

Example 12.5. Take $X = \mathbb{R}^2$ and

$$M = \mathbb{Q}^2 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1, \xi_2 \in \mathbb{Q}\}$$

Then $\overline{\mathbb{Q}^2} = \mathbb{R}^2$, since every point of $\mathbb{R}^2$ is a limit point of $\mathbb{Q}^2$:

$$\forall x_0 \in \mathbb{R}^2 \quad \forall \varepsilon > 0 \quad \exists x \in \mathbb{Q}^2 : \; x \in B_\varepsilon(x_0).$$

Exersice 12.6 Propose a metric space X and a ball $B_r(x_0)$ in X such that its closure

$$\overline{B_r(x_0)} \neq \overline{B}_r(x_0) = \{x \in X : d(x,x_0) \leq r\}$$

Def 12.7. • A subset M of X is called **dence** in X if

$$\overline{M} = X.$$

• X is called **separable** if there exists a countable subset of M which is dence.

Example 12.8 According to Example 12.5, the metric space $\mathbb{R}^2$ is separable.

Remark 12.9. A metric space is separable if there exists a countable set $M \subseteq X$ such that every ball $B_r(x)$, $r>0$, $x \in X$ contains points from M, that is,

$$\forall x \in X, \ \forall r>0 \quad B_r(x) \cap M \neq \emptyset.$$

Remark 12.10 The spaces: $\mathbb{R}$, $\mathbb{R}^n$, c, $C[a,b]$, ℓ^p, ℓ^p_n, L_p are separable,
The spaces $B[a,b]$ and ℓ^∞ are not separable

Example 12.11 We show that ℓ^p is separable.

Take $M = \{x \in \ell^p : x = (\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots),$
$\xi_k \in \mathbb{Q},\ k = 1, \dots, n,\ n \geq 1\}$

Remark that M is countable.
Indeed, we can identify

$$M_n = \{x \in \ell^p : x = (\xi_1, \dots, \xi_n, 0, 0 \dots), \xi_k \in \mathbb{Q}\}$$

with $\mathbb{Q}^n$, that is countable.

Consequently $M = \bigcup_{n=1}^{\infty} M_n$ is countable.

Let us show that $\overline{M} = X$.
By Remark 12.9, we need to take arbitrary $x \in \ell^p$, $\varepsilon > 0$ and find $y \in M$ such that $y \in B_\varepsilon(x) \Leftrightarrow d(x,y) < \varepsilon$.
Since $x \in \ell^\infty$,

$$\sum_{k=1}^{\infty} |\xi_k|^p < +\infty.$$

There exists $n \geq 1$ such that

$$\sum_{k=n+1}^{\infty} |\xi_k|^p < \delta_1 = \frac{\varepsilon^p}{2}$$

Next, we choose $\eta_k \in \mathbb{Q}$, $k = 1, \dots, n$

such that $\quad |\xi_k - \eta_k| < \delta_2 = \frac{\varepsilon}{\sqrt[p]{2n}}, \quad k = 1, \dots, n$

Take $\quad y = (\eta_1, \dots, \eta_n, 0, 0, \dots) \in M.$

Then
$$d^p(x,y) = \sum_{k=1}^{\infty} |\xi_k - \eta_k|^p =$$
$$= \sum_{k=1}^{n} |\xi_k - \eta_k|^p + \sum_{k=n+1}^{\infty} |\xi_k|^p <$$
$$< n\delta_2^p + \delta_1 = \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p$$
$$\Rightarrow \quad n\delta_2^p = \frac{\varepsilon^p}{2} \Rightarrow \quad \delta_2 = \sqrt[p]{\frac{\varepsilon^p}{n2}}, \quad \text{and}$$
$$\delta_1 = \frac{\varepsilon^p}{2}.$$

2. Convergence, Cauchy sequence, Completeness.

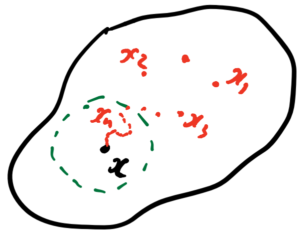
Def 12.12. A sequence $\{x_n\}_{n\geq 1}$ in a metric space $X = (X, d)$ is said to **converge** or to be **convergent** if there exists $x \in X$ such that
$$\lim_{n\to\infty} d(x_n, x) = 0.$$

- x is called the **limit of $\{x_n\}_{n\geq 1}$** and we write
$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x.$$

Remark 12.13. $\quad x_n \to x \quad$ iff

$\forall \varepsilon > 0 \quad \exists N \quad$ s.t. $\quad \forall n \geq N$

$$d(x_n, x) < \varepsilon \quad (\text{or } x_n \in B_\varepsilon(x))$$



A set M is bounded if it is contained in a ball $B_r(x_0)$, that is, $\exists\, x_0 \in X$, $r > 0$ s.t.

$$M \subseteq B_r(x_0)$$

Lemma 12.14 Let $X = (X, d)$ be a metric space. Then

(a) A convergent sequence in X is bounded and its limit is unique

(b) if $x_n \to x$ and $y_n \to y$ in X, then $d(x_n, y_n) \to d(x, y)$.