11. Metric spaces.

1. Definition and examples.

Let X be an abstract set.

Def 11.1 A metric space is a pair (X, d), where X is a set and d is a metric (or distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have $\forall x, y, z \in X$

(M1) $d(x,y) \in [0, +\infty)$

(M2) $d(x,y) = 0 \iff x = y$

(M3) $d(x,y) = d(y,x)$ (symmetry)

(M4) $d(x,y) \le d(x,z) + d(z,y)$ (triangle inequality)

Examples of metric spaces

1) Real line $\mathbb{R}$

$X = \mathbb{R}$, $d(x,y) = |x-y|$, $x, y \in \mathbb{R}$.

2) Euclidean space $\mathbb{R}^n$

$X = \mathbb{R}$, $d(x,y) = \left(\sum_{k=1}^{n} (\xi_k - \eta_k)^2 \right)^{\frac{1}{2}}$,

$x = (\xi_k)_{k=1}^{n}$, $y = (\eta_k)_{k=1}^{n}$.

3) Sequence space ℓ^∞

$$X = \ell^\infty := \{ x = (\xi_k)_{k=1}^\infty : \xi_k \in \mathbb{R},\ x \text{ is bounded} \}$$

$$d(x,y) = \sup_{k \in \mathbb{N}} |\xi_k - \eta_k|,$$

$$x = (\xi_k)_{k=1}^\infty,\ y = (\eta_k)_{k=1}^\infty.$$

4) Space c

$$c = \{ x = (\xi_k)_{k=1}^\infty : \xi_k \in \mathbb{R},\ \{\xi_k\}_{k \geq 0} \text{ converges} \}$$

$$d(x,y) = \sup_{k \in \mathbb{N}} |\xi_k - \eta_k|$$

Remark that c is a subspace of ℓ^∞ because $c \subseteq \ell^\infty$ and the metric on c is just the restriction of metric on ℓ^∞.

5) Space $B(A)$

Let $B(A)$ be the set of all bounded functions on A

$$d(x,y) := \sup_{t \in A} |x(t) - y(t)|,\quad x,y \in B(A).$$

Let us prove that $(B(A), d)$ is a metric space.

(M1): $d(x,y) \geq 0$ – trivial

(M2): $d(x,y)=0 \iff \sup_{t\in A} |x(t)-y(t)|=0$

$\iff x(t)=y(t) \quad \forall t \in [a,b]$

(M3): $d(x,y)= \sup_{t\in A} |x(t)-y(t)| =$

$= \sup_{t\in A} |y(t)-x(t)| = d(y,x)$

(M4): $d(x,y) = \sup_{t\in A} |x(t)-y(t)| =$

$= \sup_{t\in A} |x(t)-z(t)+z(t)-y(t)| =$

$\leq \sup_{t\in A} |x(t)-z(t)| + \sup_{t\in A} |z(t)-y(t)|,$

$= d(x,z) + d(z,y).$

6) Function space $C[a,b]$

X is the set of all continuous functions from $[a,b]$ to $\mathbb{R}$

$$d(x,y) = \max_{t\in[a,b]} |x(t)-y(t)|$$

$(C[a,b], d)$ is a metric subspace of $(B([a,b]), d)$.

7) Space ℓ^p, $p \geq 1$

ℓ^p is the set of all sequences $x = (\xi_k)_{k=1}^{\infty}$ in $\mathbb{R}$ such that

$$\sum_{k=1}^{\infty} |\xi_k|^p < +\infty.$$

Define $d(x,y) = \left(\sum_{k=1}^{\infty} |\xi_k - \eta_k|^p \right)^{\frac{1}{p}}$ (11.1)

we want to prove that (ℓ^p, d) is a metric space. For this we need the following inequalities:

Let $x = (\xi_k)_{k=1}^{\infty}$, $y = (\eta_k)_{k=1}^{\infty}$

Hölder inequality:

$$\sum_{k=1}^{\infty} |\xi_k \eta_k| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |\eta_k|^q \right)^{\frac{1}{q}},$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, if $p=2$, then $q=2$ and we get

Cauchy - Schwarz inequality

$$\sum_{k=1}^{\infty} |\xi_k \eta_k| \le \left(\sum_{k=1}^{\infty} |\xi_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |\eta_k|^2\right)^{\frac{1}{2}}.$$

Minkowski inequality

$$\left(\sum_{k=1}^{\infty} |\xi_k + \eta_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |\eta_k|^p\right)^{\frac{1}{p}},$$

where $p \ge 1$, $x = (\xi_k)_{k=1}^{\infty}$, $y = (\eta_k)_{k=1}^{\infty} \in \ell^p$.

Let us show that d defined by (11.1) is a distance

(M1) - (M3) are trivial

(M4) $$d(x,y) = \left(\sum_{k=1}^{\infty} |\xi_k - \eta_k|^p\right)^{\frac{1}{p}} \le$$

$$\le \left(\sum_{k=1}^{\infty} |\xi_k - \zeta_k + \zeta_k - \eta_k|^p\right)^{\frac{1}{p}} \le$$

$$\le \left(\sum_{k=1}^{\infty} \left(|\xi_k - \zeta_k| + |\zeta_k - \eta_k|\right)^p\right)^{\frac{1}{p}} \le$$

Minkowski in.

$$\le \left(\sum_{k=1}^{\infty} |\xi_k - \zeta_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |\zeta_k - \eta_k|^p\right)^{\frac{1}{p}},$$

where $x = (\xi_k)_{k=1}^{\infty}$, $y = (\eta_k)_{k=1}^{\infty}$, $z = (\zeta_k)_{k=1}^{\infty}$.

8) Space ℓ_n^p, $p \geq 1$

$$\ell_n^p = \mathbb{R}^n, \quad d(x,y) = \left(\sum_{k=1}^{n} |\xi_k - \eta_k|^p \right)^{\frac{1}{p}}.$$

9) Space $L_p[a,b]$, $p \geq 1$

Let λ be a Lebesgue measure on $[a,b]$. We always assume that two measurable functions $x, y : [a,b] \to \mathbb{R}$ are equal each other if

$$x = y \quad \lambda\text{-a.e.}$$

$L_p[a,b]$ is space of measurable functions x on $[a,b]$ (more presicely classes of equivalences) such that

$$\int_a^b |x(t)|^p \, dt < +\infty.$$

$$d(x,y) = \left(\int_a^b |x(t) - y(t)|^p \, dt \right)^{\frac{1}{p}}.$$

10) Discrete metric space

Let X be a set. Define

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y. \end{cases}$$

(X, d) is called a discrete metric space.

2. Open and closed sets

Let (X, d) be a metric space

Def. 11.2 The sets

a) $B_r(x_0) = \{x \in X : d(x, x_0) < r\}$

is called an open ball

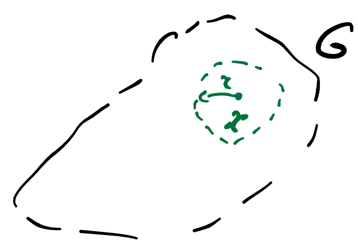
b) $\overline{B}_r(x_0) = \{x \in X : d(x, x_0) \le r\}$

is called a closed ball

with center x_0 and radius r.

Def 11.3 • A set G is called open (in X) if $\forall x \in G$ $\exists r > 0$ such that

$$B_r(x) \subset G$$



• A set F is called closed (in X) if $F^c = X \setminus F$ is open.

Exercise 11.4. a) Prove that the union of any family of open sets is open.

b) Prove that the intersection of a finite family of open sets is open.

Exercise 11.5 Show that the set

$$G = \{ x \in C[0,1] : |f(\tfrac{1}{2})| < 1 \}$$

is open in $C[0,1]$.