10. Limit theorems II. Change of variables

1. Monotone convergence theorem.

Let $(X,\mathcal{F})$ be a measurable space, λ be a measure on $\mathcal{F}$ and f, f_n be $\mathcal{F}$-measurable.

We recall that

1) $f_n \to f$ λ-a.e. $\Longleftrightarrow$ $\exists$ $\mathcal{N}\in\mathcal{F}$ s.t. $\lambda(\mathcal{N})=0$ and $\lim\limits_{n\to\infty} f_n(x) = f(x)$ $\forall x \in X\setminus\mathcal{N}$.

2) $f_n \xrightarrow{\lambda} f$ $\Leftrightarrow$ $\forall \varepsilon>0$
$$\lambda\{x\in X : |f_n(x)-f(x)| \geq \varepsilon\} \underset{n\to\infty}{\longrightarrow} 0.$$

Th. 9.10 (Monotone convergence theorem)
Let $A\in\mathcal{F}$, $f, f_n, n\geq 1$, satisfy
$$0 \leq f_n(x) \leq f_{n+1}(x), \quad \forall n\geq 1, x\in A,$$
and $f_n \to f$ λ-a.e. on A.

Then
$$\lim_{n\to\infty} \int_A f_n \, d\lambda = \int_A f \, d\lambda.$$

2. Fatou's lemma

Lemma 10.1 (Fatou lemma) Let $A \in \mathcal{F}$, and functions f_n, $n \geq 1$, satisfy

$$f_n(x) \geq 0 \quad \forall x \in A.$$

Then
$$\int_A \varliminf_{n\to\infty} f_n(x)\,\lambda(dx) \leq \varliminf_{n\to\infty} \int_A f_n\,d\lambda.$$

Remark 10.2 Fatou's lemma implies that if $f_n \geq 0$ on A, $f_n \to f$ λ-a.e. on A and $\int_A f_n\,d\lambda \leq C$ $\forall n \geq 1$, then

$$f \in L(A, \lambda) \quad \text{and} \quad \int_A f\,d\lambda \leq C.$$

To see this, just apply Fatou's lemma for the set $A \setminus \mathcal{N}$, where

$$\mathcal{N} = \{x : f_n(x) \not\to f(x)\}.$$

and properties 1), 8) of the integral

Proof of Fatou's lemma.

Consider

$$g_n(x) := \inf_{k \geq n} f_k(x), \quad x \in A, \; n \geq 1.$$

Then $0 \leq g_n(x) \leq g_{n+1}(x) \quad \forall x \in A, \; n \geq 1$

Moreover, $$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \inf_{k \geq n} f_n(x) =$$

$$= \varliminf_{n\to\infty} f_n(x).$$

We also have, $\quad g_n(x) \leq f_n(x) \quad \forall x \in A, \; n \geq 1.$

Thus $$\int_A g_n(x) \leq \int_A f_n(x).$$

By the monotone convergence theorem,

$$\lim_{n\to\infty} \int_A g_n \, d\lambda = \int_A \lim_{n\to\infty} g_n \, d\lambda =$$

$$= \int_A \varliminf_{n\to\infty} f_n \, d\lambda$$

Hence

$$\varliminf_{n\to\infty} \int_A f_n \, d\lambda \geq \varliminf_{n\to\infty} \int_A g_n \, d\lambda =$$

$$= \int_A \varliminf_{n\to\infty} f_n \, d\lambda .$$

$\blacksquare$

3. The dominated convergence theorem

Th 10.3 (Dominated convergence theorem)
Let $A \in \mathcal{F}$ and a sequence f_n satisfies

1) $f_n \to f$ a.e. on A

2) $\exists g \in L(A,\lambda)$: $|f_n(x)| \le g(x)$
$\forall n \ge 1$, $\forall x \in A$

Then $f, f_n \in L(A,\lambda)$, $n \ge 1$, and

$$\lim_{n\to\infty} \int_A f_n \, d\lambda = \int_A f \, d\lambda.$$

Proof We remark, that

$$-g(x) \le f_n(x) \le g(x) \quad \forall x \in A,\ n \ge 1.$$

Then $g + f_n \ge 0$ and $g - f_n \ge 0$ $\forall n$.
We can apply to those sequences Fatou's lemma:

$$\underline{\lim}_{n\to\infty} \int_A (g + f_n) \, d\lambda \ge \int_A (g+f) \, d\lambda,$$

$$\underline{\lim}_{n\to\infty} \int_A (g - f_n) \, d\lambda \ge \int_A (g-f) \, d\lambda.$$

Hence,

$$\int_A g\,d\lambda + \varliminf_{n\to\infty} \int_A f_n\,d\lambda \geq \int_A g\,d\lambda + \int_A f\,d\lambda$$

and

$$\int_A g\,d\lambda - \varlimsup_{n\to\infty} \int_A f_n\,d\lambda \geq \int_A g\,d\lambda - \int_A f\,d\lambda$$

Hence,

$$\int_A f\,d\lambda \leq \varliminf_{n\to\infty} \int_A f_n\,d\lambda \leq \varlimsup_{n\to\infty} \int_A f_n\,d\lambda \leq \int_A f\,d\lambda.$$

$\blacksquare$

Corollary 10.4 The claim of Th. 10.3 remains true if condition 1) is replaced by

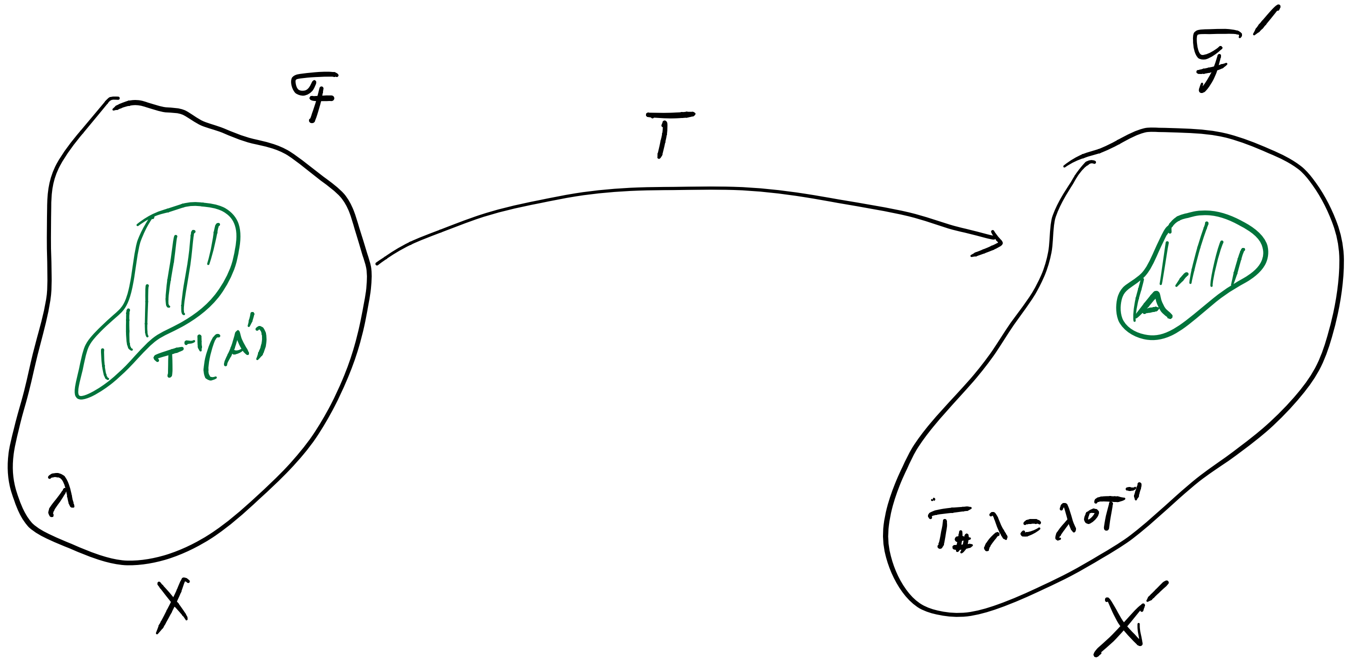
1') $f_n \xrightarrow{\lambda} f$ on A i.e.

$$\forall \varepsilon > 0 \quad \lambda\{x \in A : |f_n(x) - f(x)| \geq \varepsilon\} \underset{n\to\infty}{\longrightarrow} 0.$$

Exercise 10.5 Using Th 9.8, prove Corollary 10.4.

4. Change of variables.

We consider two measurable spaces $(X, \mathcal{F})$ and $(X', \mathcal{F}')$. Let λ be a measure on X and T be $(\mathcal{F}, \mathcal{F}')$-measurable map



We define a new measure on X' which is a push forward of the measure λ.

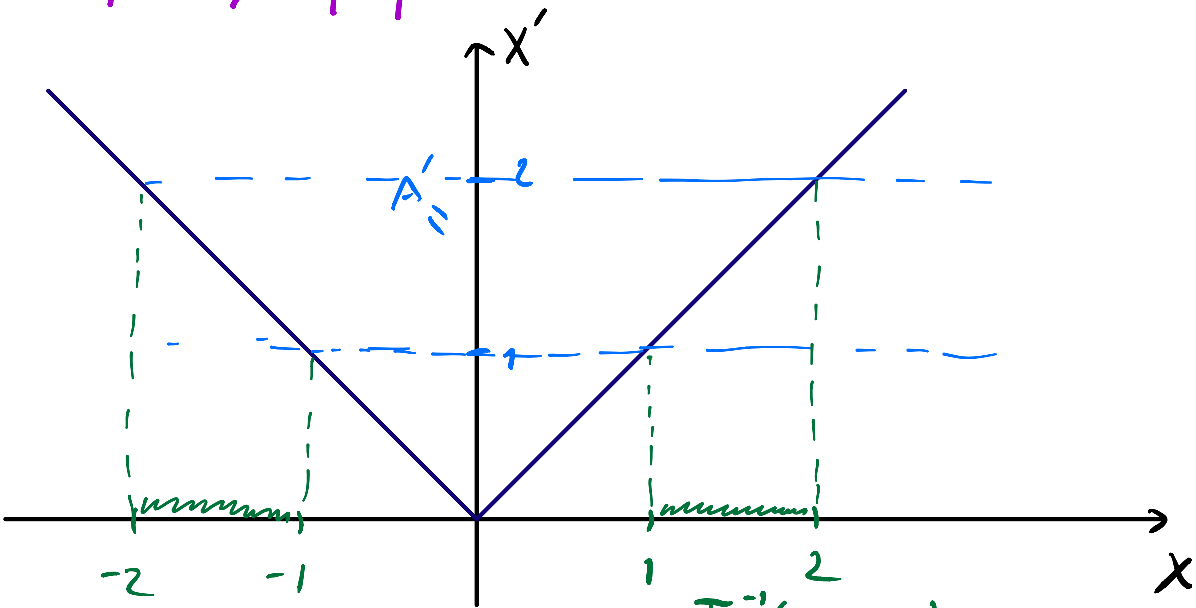
$$T_\# \lambda (A') := \lambda(T^{-1}(A')) =$$

$$= \lambda(\{x \in X : T(x) \in A'\}), \quad \forall A \in \mathcal{F}'.$$

We will also use the notation $\lambda \circ T^{-1} := T_\# \lambda$.

Exercise 10.6 Check that $T_\# \lambda$ is a measure on $\mathcal{F}'$.

Example 10.7 Let $X=\mathbb{R}$, $X'=[0,+\infty)$ and λ be the Lebesgue measure on $X=\mathbb{R}$.
Take $T(x)=|x|$



$T^{-1}([1,2]) = [1,2] \cup [-2,-1]$

Then $\underset{A'}{\lambda \circ T^{-1}([1,2])} = \lambda([1,2] \cup [-2,1]) =$
$= 2\lambda([1,2]) = 2.$

It is easy to check that $\lambda \circ T^{-1}(A') = 2\lambda(A')$

Theorem 10.7 (change of variables)
Let $f: X' \to \mathbb{R}$ be $\mathcal{F}'$-measurable function.
Then

$$\int_X f(T(x))\,\lambda(dx) = \int_{X'} f(y)\,\lambda\circ T^{-1}(dy)$$

holds, if at least of the integrals exists.

5. Comparison of Lebesgue and Riemann integrals.

Let $X = [a,b]$, $\mathcal{F} = \mathcal{B}([a,b]) = \mathcal{B}(\mathbb{R}) \cap [a,b]$ and λ be the Lebesgue measure on $\mathcal{F}$. We will denote the Lebesgue integral over λ as follows

$$\int_a^b f \, d\lambda := \int_a^b f(x)\,dx := \int_{[a,b]} f \, d\lambda$$

We denote by $R([a,b])$ the set of all Riemann integrable functions $f : [a,b] \to \mathbb{R}$ on $[a,b]$.

Th 10.8 If $f \in R([a,b])$, then $f \in L([a,b], \lambda)$ and

$$\int_a^b f(x)\,dx = \int_a^b f \, d\lambda$$

it is the Riemann integral

it is the Lebesgue integral

6. Lebesgue - Stieltjes integral.

Let $X=\mathbb{R}$, $\mathcal{H}=\{(a,b] : -\infty<a<b<+\infty\}\cup\{\emptyset\}$
$F:\mathbb{R}\to\mathbb{R}$ be non-decreasing function
We set

$$\lambda_F(\emptyset):=0,\quad \lambda_F((a,b])=F(b)-F(a),\ (a,b]\in\mathcal{H}.$$

Th 10.9. λ_F is a measure on $\mathcal{H}$.

Consequently λ_F can be extended to a measure on $r(\mathcal{H})$. We will denote this extension also by λ_F.
Next, let λ_F^* be the outer measure on 2^X generated by λ_F. Consider the class S_F of all λ_F^*-measurable sets from 2^X. By the Caratheodory theorem,

1) S_F is a σ-algebra

2) λ_F^* is a measure on S_F.

We denote this measure by λ_F.

Next, by Th 4.12, $\mathcal{H}\subset r(\mathcal{H})\subset S_F$

we can conclude that $\mathcal{B}(\mathbb{R}) = \sigma(H) \subset S_F$.
Hence, λ_F is defined on $\mathcal{B}(\mathbb{R})$.

Def. 10.10 The integral $\int_A f \, d\lambda_F$ is called the **Lebesgue-Stieltjes integral on $\mathbb{R}$** and is denoted by

$$\int_A f(x) \, dF(x) := \int_A f \, d\lambda_F .$$

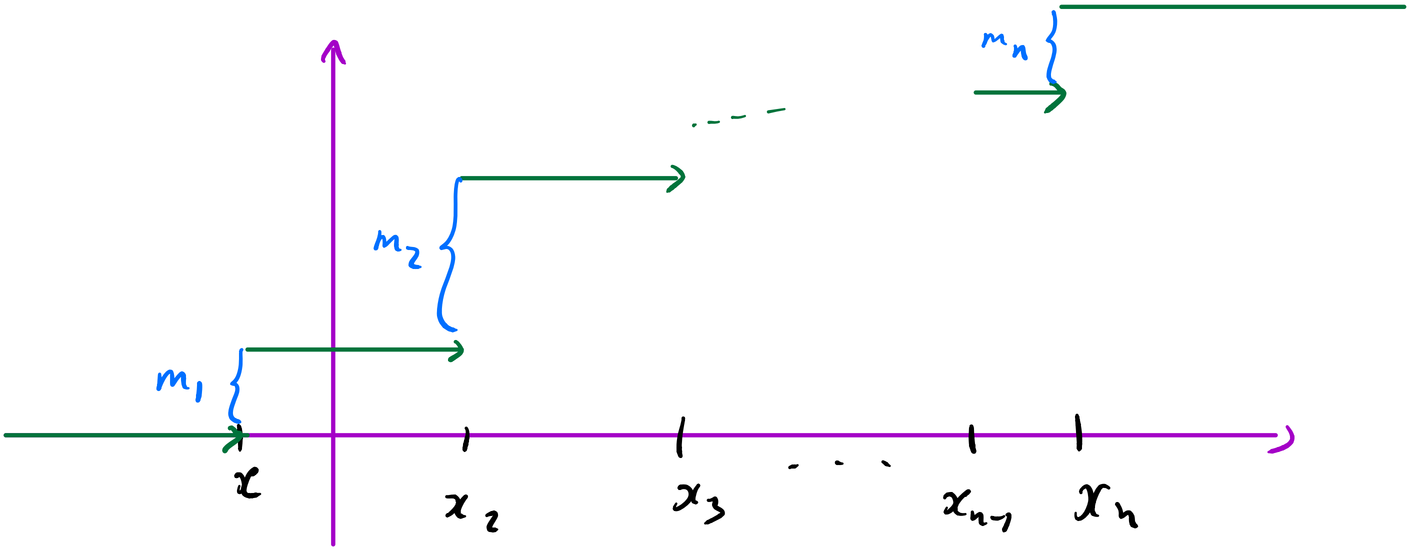
If $A = [a, b]$, then we write

$$\int_a^b f(x) \, dF(x).$$

Exercise 10.11. a) Let F is a continuously differentiable function and $F'(x) = f(x)$, $x \in \mathbb{R}$. Show that

$$\int_{-\infty}^{+\infty} g(x) \, dF(x) = \int_{-\infty}^{+\infty} g(x) f(x) \, dx$$

b) Let $x_1 < \ldots < x_n$ and $m_1, \ldots, m_n \geq 0$.
Define



Show that

$$\int_{-\infty}^{+\infty} g \, dF = \sum_{k=1}^{n} g(x_k) m_k .$$