9. Limit theorems for Lebesgue integral

1. Convergence of functions

In this lecture, we always assume that $(X, \mathcal{F})$ is a measurable space, λ is a measure on $\mathcal{F}$, and $f, f_n, n \geq 1$, are $\mathcal{F}$-measurable.

Def 9.1 Let $f, f_n : X \to \mathbb{R}, n \geq 1$, be $\mathcal{F}$-measurable functions. The sequence $\{f_n\}_{n\geq 1}$ converges to f λ-a.e. (or a.s. with respect to λ) if

$$\exists\ \mathcal{N} \in \mathcal{F} \text{ s.t. } \lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in X \setminus \mathcal{N}, \quad \text{and } \lambda(\mathcal{N}) = 0.$$

Notation: $f_n \to f$ λ-a.e

Exercise 9.2 Let $f_n \to f$ λ-a.e. and $f_n \to g$ λ-a.e. Show that $f = g$ a.e.

Def 9.3 Let $f, f_n : X \to \mathbb{R}, n \geq 1$, be $\mathcal{F}$-measurable functions. The sequence $\{f_n\}_{n\geq 1}$ converges to f in measure if

$$\forall \varepsilon > 0 \quad \lambda\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\} \to 0, \quad n \to \infty$$

Notation: $f_n \xrightarrow{\lambda} f$.

Theorem 9.4 Let $f_n \xrightarrow{\lambda} f$ and $f_n \xrightarrow{\lambda} g$. Then $f = g$ λ-a.e.

Proof We first remark

$$\{x : |f(x) - f_n(x) + f_n(x) - g(x)| \geq \varepsilon\} \subseteq$$

$$\subseteq \{x : |f(x) - f_n(x)| \geq \tfrac{\varepsilon}{2}\} \cup \{x : |f_n(x) - g(x)| \geq \tfrac{\varepsilon}{2}\}$$

Hence

$$\forall \varepsilon > 0 \quad \lambda(\{x : |f(x) - g(x)| \geq \varepsilon\}) =$$

$$= \lambda(\{x : |f(x) - f_n(x) + f_n(x) - g(x)| \geq \varepsilon\}) \leq$$

$$\leq \lambda(\{x : |f(x) - f_n(x)| \geq \tfrac{\varepsilon}{2}\}) +$$

$$+ \lambda(\{x : |f_n(x) - g(x)| \geq \tfrac{\varepsilon}{2}\}) \to 0, \; n \to \infty$$

Thus, $\forall \varepsilon > 0 \quad \lambda(\{x : |f(x) - g(x)| \geq \varepsilon\}) = 0$.

Next

$$\{x : f(x) \neq g(x)\} = \bigcup_{k=1}^{\infty} \{x : |f(x) - g(x)| \geq \tfrac{1}{k}\}$$

By the σ-semiadditivity of the measure λ

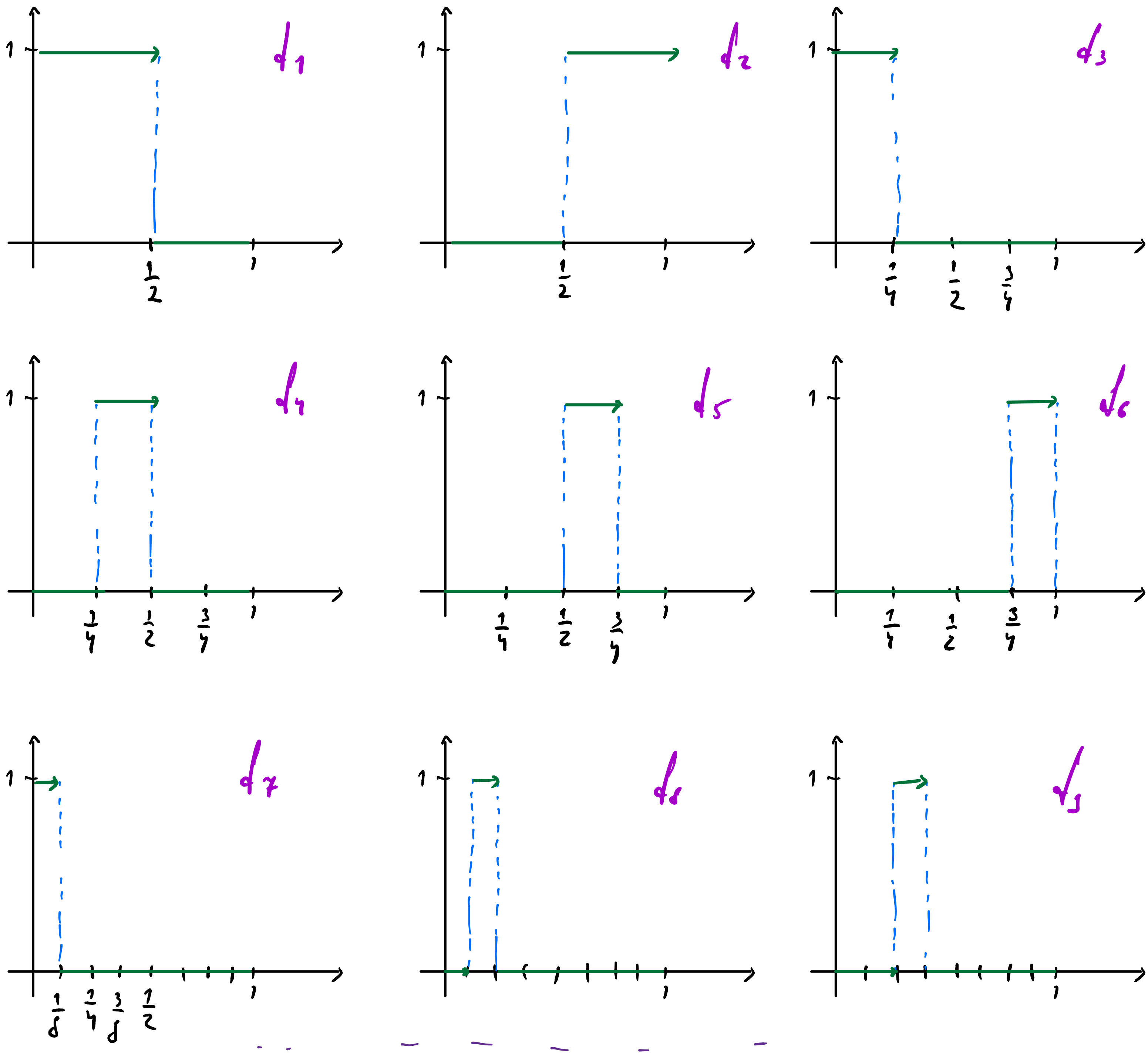
$$\lambda(\{x : f(x) \neq g(x)\}) = 0.$$

∎

Convergence in measure does not imply convergence λ-a.e. Even it does not imply the convergence in some fixed point x.

Example 9.5 Take the following sequence

$X = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, λ - Lebesgue measure

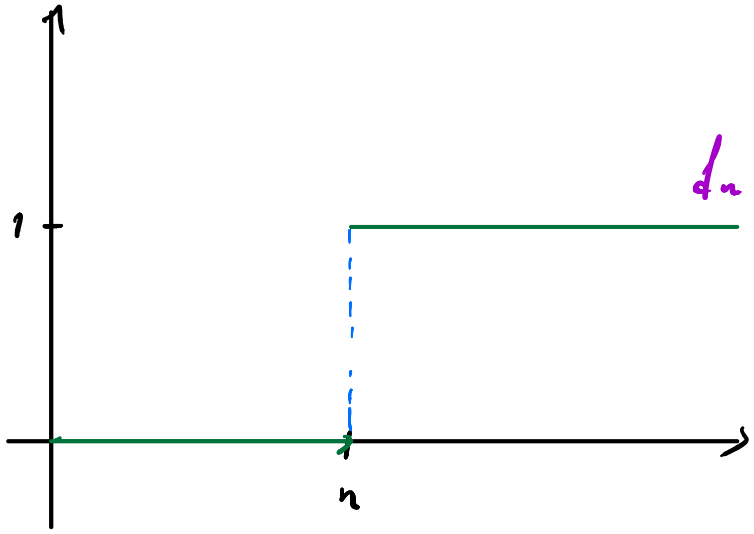


Then $f_n \xrightarrow{\lambda} f$ but $f_n \to f$ λ-a.e.
Moreover, $\forall x \in [0,1]$ $f_n(x) \not\to f(x)$.

Convergence λ-a.e. does not imply convergence in measure

Example 9.6. We take $X = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$,
λ - Lebesgue measure

Take $f_n(x) = \mathbb{I}_{[n, +\infty)}(x)$, $x \in \mathbb{R}$.



Then $\forall x \in \mathbb{R}$ $f_n(x) \to 0$ $\Rightarrow f \to 0$ λ-a.e.
But $f_n \overset{\lambda}{\not\to} f$, since

$\varepsilon < 1$

$$\lambda(\{x : |f_n(x) - f(x)| > \varepsilon\}) = \lambda([n, +\infty)) = $$
$$= +\infty.$$

Th. 9.7 (Lebesgue) Let $f, f_n : X \to \mathbb{R}$, $n \geq 1$, be $\mathcal{F}$-measurable functions and $\lambda(X) < +\infty$. Then

$$f_n \to f \ \ \lambda\text{-a.e.} \implies f_n \xrightarrow{\lambda} f .$$

Proof Let $\varepsilon > 0$ be fixed. We set

$$A_n := \{ x : |f_n(x) - f(x)| \geq \varepsilon \} \in \mathcal{F}.$$

Let $B_n := \bigcup_{k=n}^{\infty} A_k \in \mathcal{F}$.

We remark, that B_n, $n \geq 1$ decreases.

Set

$$B := \bigcap_{n=1}^{\infty} B_n = \overline{\lim_{n\to\infty}} A_n = \{ x : \text{for infinite number of indices } n \ \ |f_n(x) - f(x)| > \varepsilon \}$$

Then $B \subseteq \{ x : f_n(x) \not\to f(x) \}$ and consequently $\lambda(B) = 0$, by the convergence $f_n \to f$ λ-a.e.

Moreover, $\lambda(B_1) \leq \lambda(X) < \infty$.

Then by the continuity of measure,

$$0 = \lambda(B) = \lim_{n\to\infty} \lambda(B_n).$$

So, $\lim_{n\to\infty} \lambda(A_n) \leq \lim_{n\to\infty} \lambda(B_n) = 0$. $\blacksquare$

Th 9.8 (Riesz) Let $f_n \xrightarrow{\lambda} f$. Then there exists a subsequence $\{f_{n_k}\}_{k\geq 1}$ such that $f_n \to f$ λ-a.e

Th 9.9 (Subsequence criterion) Let $\lambda(X) < +\infty$. $f_n \xrightarrow{\lambda} f$ if and only if every subsequence $\{f_{n_k}\}_{k\geq 1}$ has a subsubsequence $\{f_{n_{k_j}}\}_{j\geq 1}$ such that

$$f_{n_{k_j}} \to f \quad \lambda\text{-a.e.}$$

2. Monotone convergence theorem

Th 9.10 (Monotone convergence theorem) Let $A \in \mathcal{F}$, f, f_n, $n \geq 1$, satisfy

$$0 \leq f_n(x) \leq f_{n+1}(x), \quad \forall n \geq 1, \ x \in A,$$

and $f_n \to f$ λ-a.e. on A.

Then
$$\lim_{n\to\infty} \int_A f_n \, d\lambda = \int_A f \, d\lambda.$$

Proof We first remark that by monotonisity of f_n, we have

$$(8.1) \quad \int_A f_1 \, d\lambda \le \int_A f_2 \, d\lambda \le \dots \le \int_A f_n \, d\lambda \le \int_A f_{n+1} \, d\lambda \le \dots \int_A f \, d\lambda$$

Since we have increasing sequence of numbers, there exists the limit

$$\alpha := \lim_{n \to \infty} \int_A f_n \, d\lambda \le +\infty.$$

We can assume that $\alpha < +\infty$.

Otherwise, the equality

$$\int_A f \, d\lambda = \lim_{n \to \infty} \int_A f_n \, d\lambda$$

trivially follows from (8.1).

Since $\alpha < +\infty$, $\int_A f_n \, d\lambda < +\infty$, $\forall n \ge 1$.

We take a simple $\mathcal{F}$-measurable function

$$p \in K(f)$$

and $c \in (0,1)$, and set

$$A_n := \{ x \in A : f_n(x) \ge c\, p(x) \} \in \mathcal{F}$$

We know that

1) $A_n \subseteq A_{n+1}$. Indeed,

Take $x \in A_n \Rightarrow f_n(x) \geq c\,p(x) \Rightarrow$

$\Rightarrow f_{n+1}(x) \geq f_n(x) \geq c\,p(x)$

$\Rightarrow x \in A_{n+1}$

2) $\bigcup_{n=1}^{\infty} A_n = A$.

Since $A_n \subseteq A$, we have $\bigcup_{n=1}^{\infty} A_n \subseteq A$.

Next, take $x \in A$. Remark

$$c\,p(x) < p(x) \leq f(x)$$

Since $f_n(x) \to f(x)$, $\exists\, n$ s.t.

$$c\,p(x) \leq f_n(x) \leq f(x)$$

$\Rightarrow x \in A_n$. Hence, $A \subseteq \bigcup_{n=1}^{\infty} A_n$.

This finishes the proof of 2).

By properties 7) and 3)

$$\int_A f_n \, d\lambda \overset{7)}{\geq} \int_{A_n} f_n \, d\lambda \overset{3),5}{\geq} c \int_{A_n} p \, d\lambda$$

Hence $c \int_{A_n} p \, d\lambda \leq \int_A f_n \, d\lambda \leq \alpha$

By the σ-additivity of the integral (prop. 12) and the continuity of a measure

$$c \int_A p \, d\lambda = \lim_{n \to \infty} c \int_{A_n} p \, d\lambda \leq \alpha$$

Since $$c \int_A p \, d\lambda \leq \alpha \qquad \forall p \in K(f)$$

$$c \int_A f \, d\lambda = c \cdot \sup_{p \in K(f)} \int_A p \, d\lambda = \sup_{p \in K(f)} c \int_A p \, d\lambda \leq \alpha .$$

So, $c \int_A f \, d\lambda \leq \alpha$, where $c \in (0,1)$

Sending $c \to 1-$, we get

$$\int_A f \, d\lambda \leq \alpha$$

$\blacksquare$