8. Properties of Lebesgue integral

1. Basic properties.

Let $(X, \mathcal{F})$ be a measurable space and $f: X \to \mathbb{R}$ be an $\mathcal{F}$-measurable function. Let also $A \in \mathcal{F}$. We recall the definition of the Lebesgue integral

Part I. Let f be a ($\mathcal{F}$-measurable) simple function

$$f(x) = \sum_{k=1}^{m} a_k \mathbb{I}_{A_k},$$

$a_1, \dots, a_m$ are distinct numbers from $\mathbb{R}$

$$A_k = f^{-1}(\{a_k\}) = \{x : f(x) = a_k\}$$

$$\int_A f \, d\lambda = \sum_{k=1}^{m} a_k \lambda(A \cap A_k)$$

(we assume $a_k \lambda(A \cap A_k) = 0$ as $a_k = 0$, $\lambda(A \cap A_k) = +\infty$)

Part II $f \geq 0$.

$$\int_A f \, d\lambda = \sup_{p \in K(f)} \int_A p \, d\lambda,$$

$K(f)$ is the set of all $\mathcal{F}$-measurable simple functions p s.t. $0 \leq p(x) \leq f(x)$, $x \in X$.

Part III General case.

$$\int_A f \, d\lambda = \int_A f_+ \, d\lambda - \int_A f_- \, d\lambda,$$

where

$$f_+(x) = \max\{f(x), 0\}$$
$$f_-(x) = -\min\{f(x), 0\}.$$

We always assume that $A \in \mathcal{F}$ and $f, g : X \to \mathbb{R}$ are $\mathcal{F}$-measurable.

1) if $\lambda(A) = 0$, then

$$\int_A f \, d\lambda = 0$$

2) Let $\lambda(A) < +\infty$, and $f(x) = C$, $x \in A$. Then $f \in L(A, \lambda)$ and

$$\int_A C \, d\lambda = C \lambda(A)$$

3) Let $0 \le f(x) \le g(x)$, $x \in A$. if $g \in L(A, \lambda)$, then $f \in L(A, \lambda)$ and

$$\int_A f\,d\lambda \le \int_A g\,d\lambda$$

The proof follows from Def 7.5 Part II and the fact that $K(f) \subseteq K(g)$. So

$$\sup_{p \in K(f)} \int p\,d\lambda \le \sup_{p \in K(g)} \int p\,d\lambda < +\infty .$$

4) Let $A \neq \emptyset$, $\lambda(A) < +\infty$, f be bounded on A. Then $f \in L(A, \lambda)$ and

$$\inf_A f \cdot \lambda(A) \le \int_A f\,d\lambda \le \sup_A f\ \lambda(A)$$

5) Let $f \in L(A, \lambda)$, $c \in \mathbb{R}$. Then $cf \in L(A, \lambda)$ and

$$\int_A cf\,d\lambda = c \int_A f\,d\lambda$$

6) Let $f, g \in L(A, \lambda)$ and $f(x) \le g(x)$, $x \in A$. Then

$$\int_A f\,d\lambda \le \int_B g\,d\lambda$$

7) Let $A, B \in \mathcal{F}$, $B \subset A$ and $f \in L(A, \lambda)$. Then $f \in L(B, \lambda)$. If additionally, f is nonnegative, then

$$\int_B f \, d\lambda \leq \int_A f \, d\lambda .$$

8) Let $A, B \in \mathcal{F}$, $A \cap B = \emptyset$, and $f \in L(A, \lambda)$, $f \in L(B, \lambda)$. Then $\int_{A \cup B} f \, d\lambda = \int_A f \, d\lambda + \int_B f \, d\lambda$

9) $f \in L(A, \lambda)$ if and only if $|f| \in L(A, \lambda)$.

$\Rightarrow$) We write $f = f_+ - f_-$, $|f| = f_+ + f_-$

Remark that $f \in L(A, \lambda)$ iff

$$\int_A f_+ \, d\lambda < +\infty , \quad \int_A f_- \, d\lambda < +\infty$$

Consider the sets

$$A_- := \{ x \in A : f(x) < 0 \} \in \mathcal{F}$$

$$A_+ := \{ x \in A : f(x) \geq 0 \} \in \mathcal{F}$$

Then $A_- \cap A_+ = \emptyset$.

Hence

$$\int_A |f| \, d\lambda \overset{8)}{=} \int_{A_-} |f| \, d\lambda + \int_{A_+} |f| \, d\lambda =$$

$$= \int_{A_-} f_- \, d\lambda + \int_{A_+} f_+ \, d\lambda \overset{7)}{\leq}$$

$$\leq \int_A f_- \, d\lambda + \int_A f_+ \, d\lambda < +\infty.$$

$\Rightarrow$ $|f| \in L(A, \lambda)$.

$\Leftarrow$) Let $|f| \in L(A, \lambda)$. Since, we have on A

$$0 \leq f_- \leq |f|, \quad 0 \leq f_+ \leq |f|,$$

we have $\int_A f_- \, d\lambda < \infty$, $\int_A f_+ \, d\lambda < \infty$,

by 3).

10) Let $f \in L(A, \lambda)$ and $|g(x)| \leq f(x) \; \forall x \in A$,
Then $g \in L(A, \lambda)$ and

$$\left| \int_A g \, d\lambda \right| \leq \int_A |f| \, d\lambda$$

11) Let $f, g \in L(A, \lambda)$. Then $f+g \in L(A,\lambda)$ and

$$\int_A (f+g)\, d\lambda = \int_A f\, d\lambda + \int_A g\, d\lambda.$$

12) (σ-additivity of the integral)

Let $f \in L(X, \lambda)$. Then the function

$$\mu(A) := \int_A f\, d\lambda, \quad A \in \mathcal{F}$$

is σ-additive. In particular, if $f \geq 0$, then μ is a measure on $\mathcal{F}$.

Def. 8.1 We say that $f = g$ λ-a.e. or a.e. (almost everywhere) on A if

$$\lambda(\{x \in A : f(x) \neq g(x)\}) = 0.$$

Example 8.2

The functions $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$, $x \in \mathbb{R}$, and $g(x) = 0$, $x \in \mathbb{R}$

are equal a.e.

Remark that the set $\{x \in A : f(x) \neq g(x)\} \in \mathcal{F}$, since

$$\{x \in A : f(x) \neq g(x)\} = \{x \in A : f(x) - g(x) \neq 0\}$$

$$= (f-g)^{-1}(\underbrace{\mathbb{R} \setminus \{0\}}_{\in \mathcal{B}(\mathbb{R})}) \in \mathcal{F},$$

and $f-g$ is $\mathcal{F}$-measurable as the difference of two measurable functions.

13) Let $g = f$ a.e. on A and $f \in L(A, \lambda)$. Then $g \in L(A, \lambda)$ and

$$\int_A f \, d\lambda = \int_A g \, d\lambda$$

14) Let $f \in L(A, \lambda)$, $f \geq 0$. If $\int_A f \, d\lambda = 0$ then $f = 0$ a.e. on A.

2. Converges of functions

Def 8.3 Let $f, f_n : X \to \mathbb{R}$, $n \geq 1$, be $\mathcal{F}$-measurable functions. The sequence $\{f_n\}_{n \geq 1}$ converges to f λ-a.e.

(or a.s. with respect to λ) if

$$\exists\ \varphi \in \mathcal{F} \text{ s.t. } \lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in X \setminus \varphi$$

Notation: $f_n \to f \quad \lambda$-a.e

Exercise 8.4 Let $f_n \to f$ λ-a.e. and $f_n \to g$ λ-a.e.

Show that $f = g$ a.e.