7. Lebesgue integral

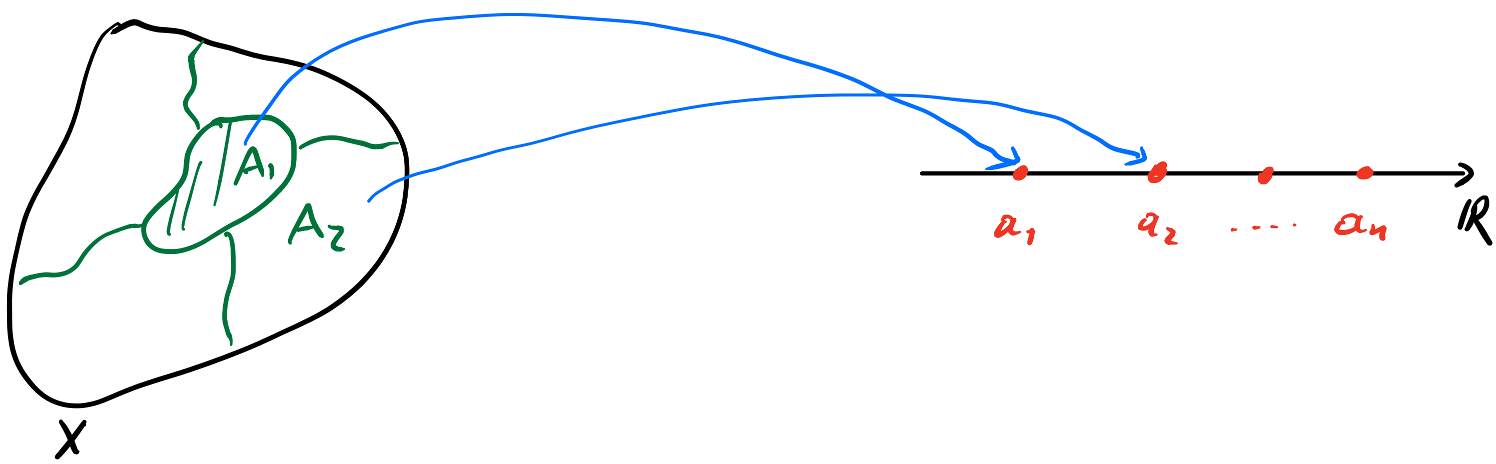
1. Approximation by simple functions.

Let $(X, \mathcal{F})$ be a measurable space, λ be a measure on $\mathcal{F}$.

Def. 7.1 A function $f: X \to \mathbb{R}$ is called *simple* if the set $f(X)$ consists of a finite number of elements, that is, there exist distinct $a_1, a_2, \dots, a_m \in \mathbb{R}$ s.t.

$$f(x) = \sum_{k=1}^{m} a_k \mathbb{I}_{A_k} \tag{7.1}$$

where $A_k = \{ x \in X : f(x) = a_k \} = f^{-1}(\{a_k\})$



Remark 7.2 The sets $A_1, \dots, A_n \in \mathcal{F}$ if and only if the function f is measurable.

Exercise 7.3 Prove that the sum and the product of simple functions are simple functions.

Th 7.4 Let f be a non-negative function ($f: X \to [0, +\infty)$). The function f is $\mathcal{F}$-measurable if and only if there exists a sequence $\{f_n\}_n$ of simple $\mathcal{F}$-measurable functions such that $\forall x \in X$ $f_n(x) \le f_{n+1}(x)$ and

$$f(x) = \lim_{n \to \infty} f_n(x)$$

Proof $\Leftarrow$) Follows from Th 6.8

$\Rightarrow$) Let f be $\mathcal{F}$-measurable. For $n \in \mathbb{N}$ we consider numbers $\frac{k}{2^n}$, $k = 0, \dots, n2^n - 1$ and define

$$A_n^k := \{ x \in X : \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n} \} \in \mathcal{F}$$

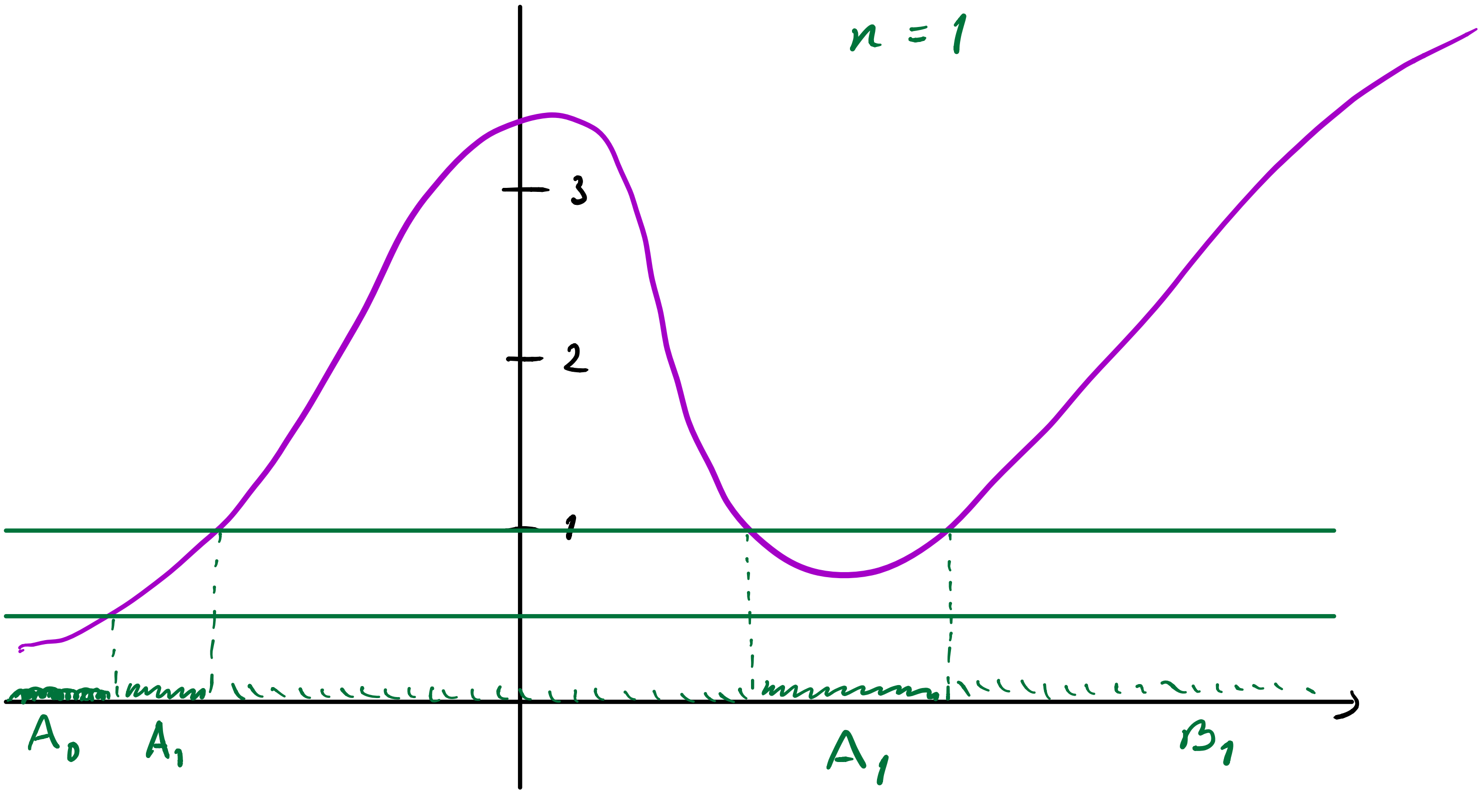
$$B_n := \{ x \in X : f(x) \ge n \} \in \mathcal{F}$$

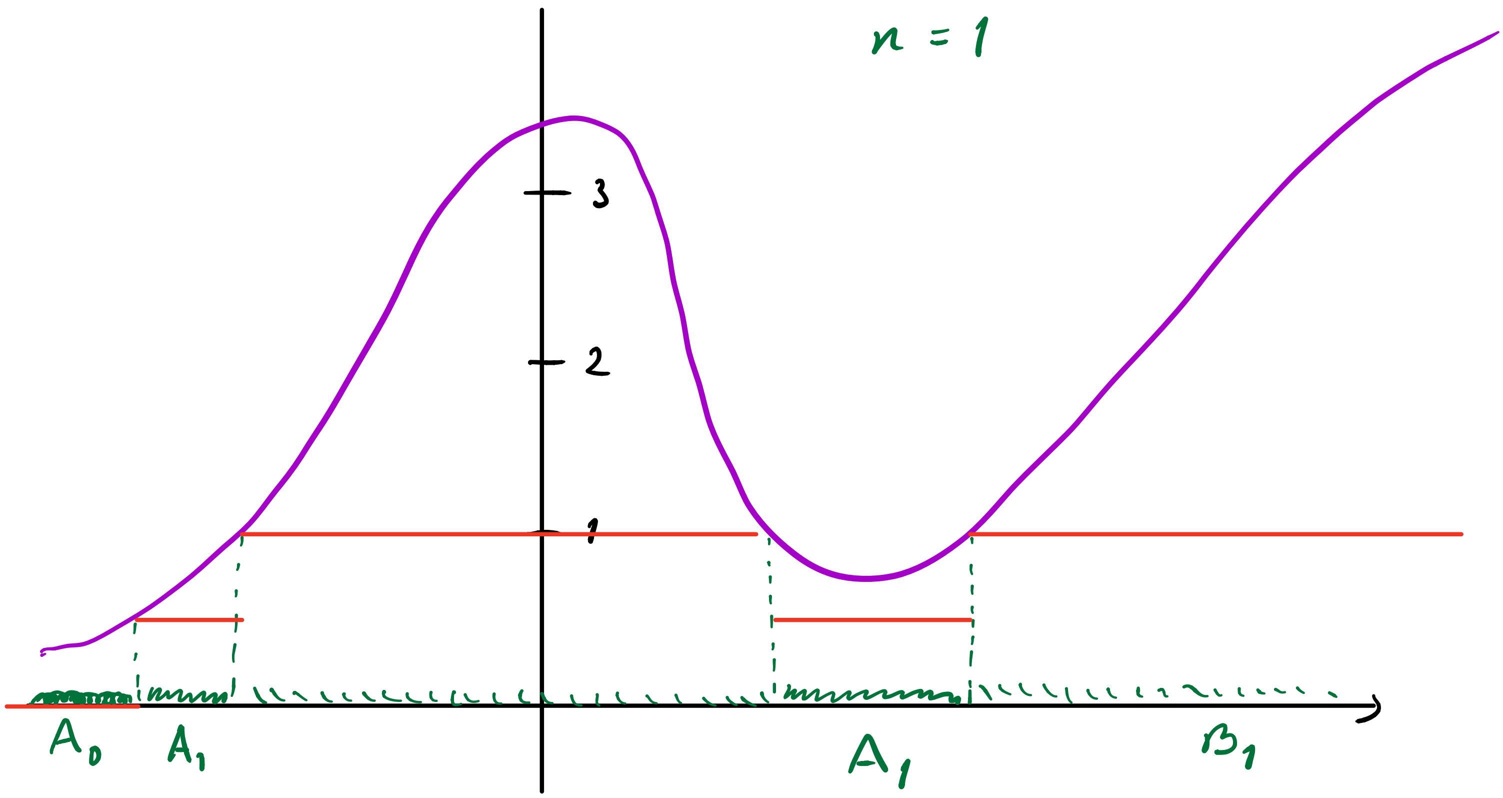
because f is $\mathcal{F}$-measurable

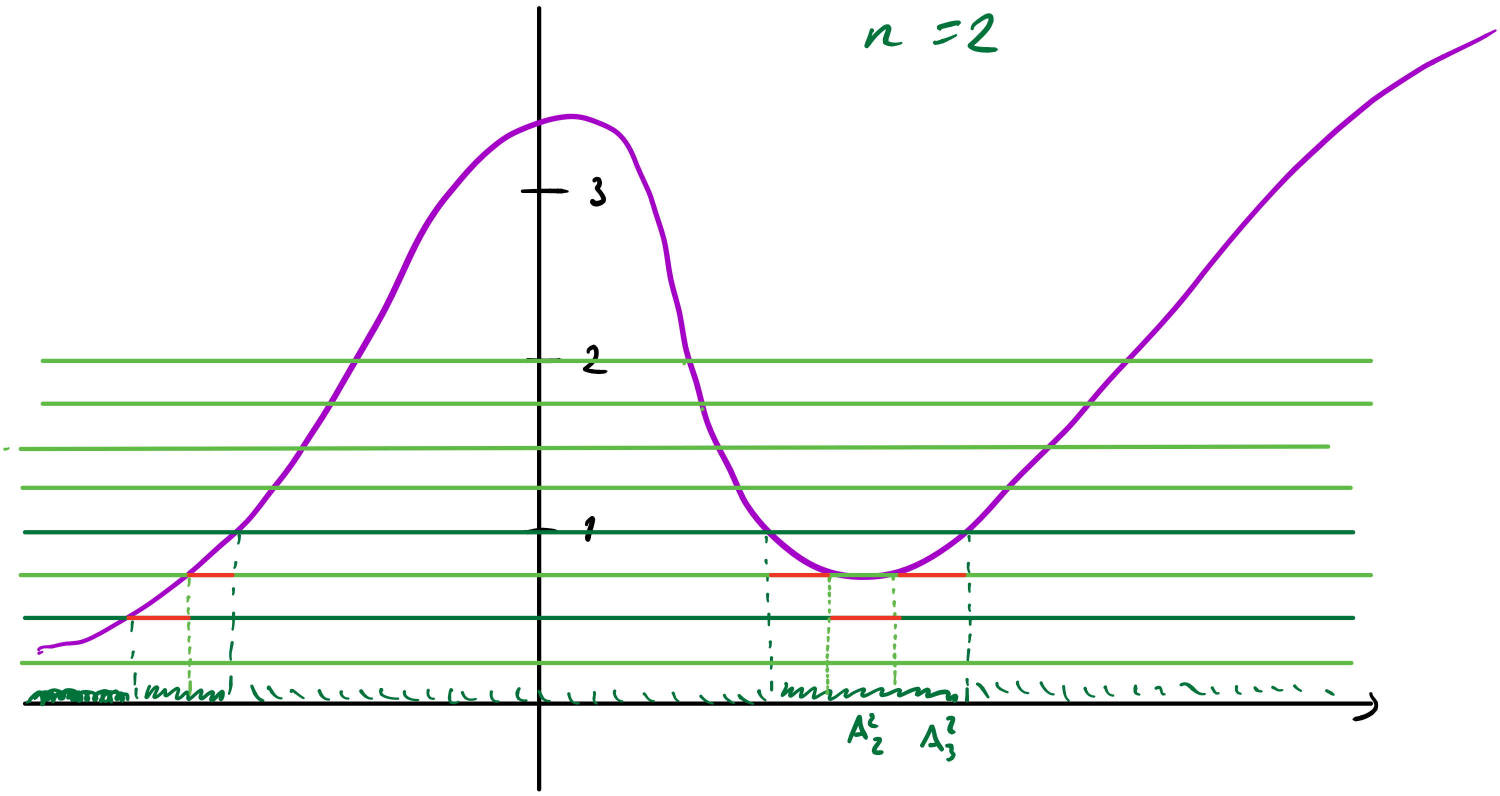
Remark: $A_n^k = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n} \right) \right)$

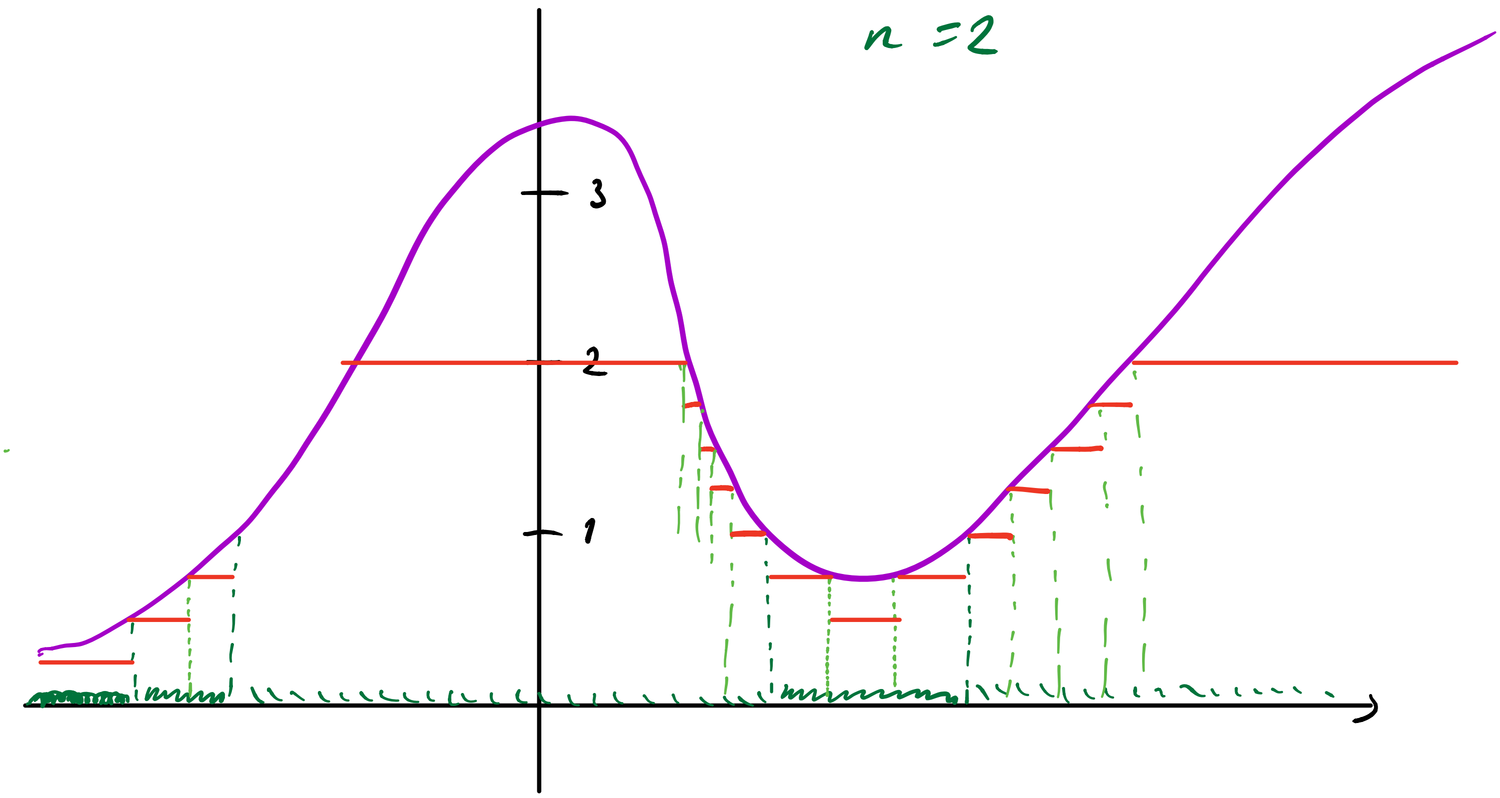
$$B_n = f^{-1}([n, +\infty))$$

Take $f_n(x) = \sum_{k=0}^{n2^n - 1} \frac{k}{2^n} \mathbb{I}_{A_n^k}(x) + n \mathbb{I}_{B_n}(x)$


$n = 1$
3
2
1
A_0
A_1
A_1
B_1


$n = 1$
3
2
1
A_0
A_1
A_1
B_1


n = 2
3
2
1
A^2_2
A^2_3


n = 2
3
2
1

2. Definition of the integral.

Def 7.5 Part I: Let f be a non-negative $\mathcal{F}$-measurable simple function defined by (7.1) and $A \in \mathcal{F}$. The value

$$\int_A f \, d\lambda := \int_A f(x) \lambda(dx) =$$

$$= \sum_{k=1}^{m} a_k \lambda(A \cap A_k)$$

is called the Lebesgue integral of f over A. Here we assume $a_k \lambda(A \cap A_k) = 0$ if $a_k = 0$, $\lambda(A \cap A_k) = +\infty$

Part II: Let $A \in \mathcal{F}$ and $f : X \to \mathbb{R}$ be $\mathcal{F}$-measurable and non-negative function. The value

$$\int_A f \, d\lambda := \int_A f(x) \, \lambda(dx) := \sup_{p \in K(f)} \int_A p(x) \, \lambda(dx)$$

is called the Lebesgue integral of f over A, where $K(f)$ is the set of all simply functions $p : X \to \mathbb{R}$ s.t.

$$0 \le p(x) \le f(x), \quad x \in X.$$

Remark 7.6 (Other way of definition of the integral in Part II) Let f_n be se sequence from Th. 7.4. which converges to f. Then

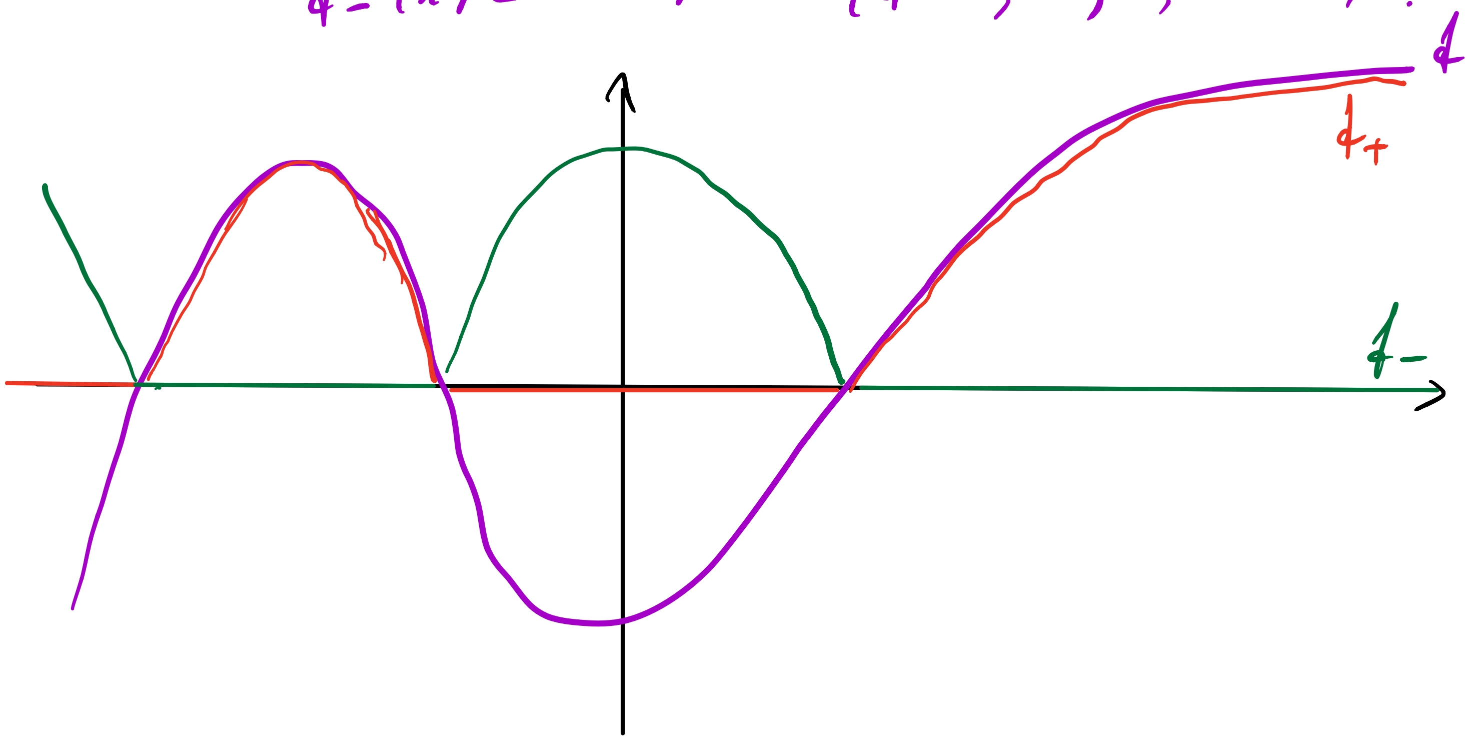
$$\int_A f(x)\,\lambda(dx) := \lim_{n\to\infty} \int_A f_n(x)\,\lambda(dx).$$

This two approaches define the same object.

Let $f: X \to \mathbb{R}$ be any function. We consider its positive and negative parts:

$$f_+(x) = \max\{f(x), 0\}, \quad x \in X.$$

$$f_-(x) = -\min\{f(x), 0\}, \quad x \in X.$$



Then trivially

$$f(x) = f_+(x) - f_-(x) \quad , x \in X$$

$$|f(x)| = f_+(x) + f_-(x) \quad , x \in X.$$

Def 7.5 Part III Let $A \in \mathcal{F}$, $f: X \to \mathbb{R}$ be $\mathcal{F}$-measurable function. If one of the integrals

$$\int_A f_+ \, d\lambda \quad \text{and} \quad \int_A f_- \, d\lambda \qquad (7.2)$$

is finite, then

$$\int_A f(x)\,dx := \int_A f\,d\lambda := \int_A f(x)\,\lambda(dx) =$$

$$= \int_A f_+\,d\lambda - \int_A f_-\,d\lambda$$

is called the Lebesgue integral of f over A.

- If two integrals (7.2) are finite then the function f is called Lebesgue integrable on A.
- The class of all Lebesgue integrable functions on A is denoted by $L(A, \lambda)$