6. Properties of measurable functions

1. One condition of measurability.

Let $(X, \mathcal{F})$, $(X', \mathcal{F}')$ be measurable spaces. We recall that f is $(\mathcal{F}, \mathcal{F}')$-measurable if

$$(6.1) \quad \forall A' \in \mathcal{F}' \quad f^{-1}(A') = \{x \in X : f(x) \in A'\} \in \mathcal{F}.$$

In general, property (6.1) is complicated to check, since the class $\mathcal{F}'$ can be too large. The next theorem says that (6.1) is enough to check only for some subclass of $\mathcal{F}'$ in the case $\mathcal{F}' = \sigma(H)$.

Th 6.1 Let $(X, \mathcal{F})$, $(X', \mathcal{F}')$ be measurable spaces and

$$\mathcal{F}' = \sigma(H), \quad H \subset 2^{X'}.$$

A map $f : X \to X'$ is $(\mathcal{F}, \mathcal{F}')$-measurable if and only if

$$\forall A' \in H \quad f^{-1}(A') \in \mathcal{F}.$$

Proof $\Rightarrow$) It follows from the definition of measurability, since

$$A' \in \mathcal{H} \Rightarrow A' \in \mathcal{F}' \Rightarrow f^{-1}(A') \in \mathcal{F}$$

$\Leftarrow$) Set $Q := \{A' \in \mathcal{F}' : f^{-1}(A') \in \mathcal{F}\}$.

Then $\mathcal{H} \subset Q \subset \mathcal{F}' = \sigma(\mathcal{H})$

Let us show that Q is a σ-algebra.

Indeed,

1) $\emptyset \in Q$ because

$$f^{-1}(\emptyset) = \emptyset \in \mathcal{F}.$$

2) $A_1', A_2', \ldots \in Q$. Then $f^{-1}(A_k') \in \mathcal{F}$

Consider $\bigcup_{k=1}^{\infty} A_k' = A'$. So,

$$f^{-1}(A') = f^{-1}\Big(\bigcup_{k=1}^{\infty} A_k'\Big) = \bigcup_{k=1}^{\infty} f^{-1}(A_k') \in \mathcal{F},$$

since $\mathcal{F}$ is a σ-algebra

3) if $A', B' \in Q$ then

$$f^{-1}(B' \setminus A') = f^{-1}(B') \setminus f^{-1}(A') \in \mathcal{F}$$

$\Rightarrow B' \setminus A' \in Q$.

Hence $\sigma(\mathcal{H}) \subset Q \Rightarrow \mathcal{F}' = \sigma(\mathcal{H}) = Q$. $\blacksquare$

Corollary 6.2. Let $f: X \to \mathbb{R}$. The following statements are equivalent.

1) f is $\mathcal{F}$-measurable;
2) $\forall a \in \mathbb{R}$, $f^{-1}((-\infty, a)) = \{x \in X : f(x) < a\} \in \mathcal{F}$
3) $\forall a \in \mathbb{R}$, $f^{-1}((-\infty, a]) = \{x \in X : f(x) \leq a\} \in \mathcal{F}$
4) $\forall a \in \mathbb{R}$, $f^{-1}((a, +\infty)) = \{x \in X : f(x) > a\} \in \mathcal{F}$
5) $\forall a \in \mathbb{R}$, $f^{-1}([a, +\infty)) = \{x \in X : f(x) \geq a\} \in \mathcal{F}$

Proof Let us only show that 1) $\Leftrightarrow$ 2) (other implications: 1) $\Leftrightarrow$ $\cdot$) are similar)

We remark that for

$$H := \{(-\infty, a), \quad a \in \mathbb{R}\}$$

we have

$$\mathcal{B}(\mathbb{R}) = \sigma(H).$$

Condition of 2) gives that $\forall A \in H$

$$f^{-1}(A) \in \mathcal{F}$$

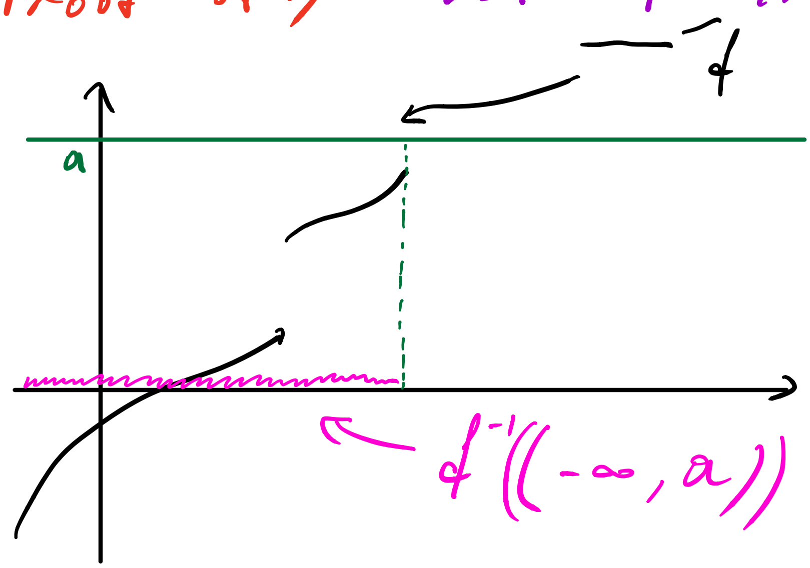
Hence, by Th. 6.1. f is $\mathcal{F}$-measurable (i.e. f is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable) if and only if it satisfies 2). $\blacksquare$

Application of corollary ($X = X' = \mathbb{R}$, $\mathcal{F} = \mathcal{F}' = \mathcal{B}(\mathbb{R})$)

1) Every monotone function $f: \mathbb{R} \to \mathbb{R}$ is Borel measurable

2) Every continuous function $f: \mathbb{R} \to \mathbb{R}$ is Borel measurable.

Proof of 1) Let f increases



$f^{-1}((-\infty, a))$ – is always an interval

$\Rightarrow f^{-1}((-\infty, a)) \in \mathcal{B}(\mathbb{R})$. ∎

Proof of 2) We know that f is continuous if and only if the preimage $f^{-1}(G)$ every open set G in $\mathbb{R}$ is open.

Consequently, $f^{-1}(\underbrace{(-\infty, a)}_{\text{open}})$ is open.

Since every open set is Borel measurable, f is a Borel measurable function. ∎

Corollary 6.3 Let $f: \mathbb{R}^d \to \mathbb{R}^m$ be continuous, then f is Borel measurable.

Proof The proof is similar to the previous proof. Let $\mathcal{H} = \{ G \subseteq \mathbb{R}^m : G\text{-open} \}$.

Then
$$\mathcal{B}(\mathbb{R}^m) = \sigma(\mathcal{H}).$$

Let $G \in \mathcal{H}$ (G is open). Then $f^{-1}(G)$ is open in $\mathbb{R}^d$ because it is continuous. Hence, $f^{-1}(G) \in \mathcal{B}(\mathbb{R}^d)$. By Th. 6.1 f is Borel measurable.

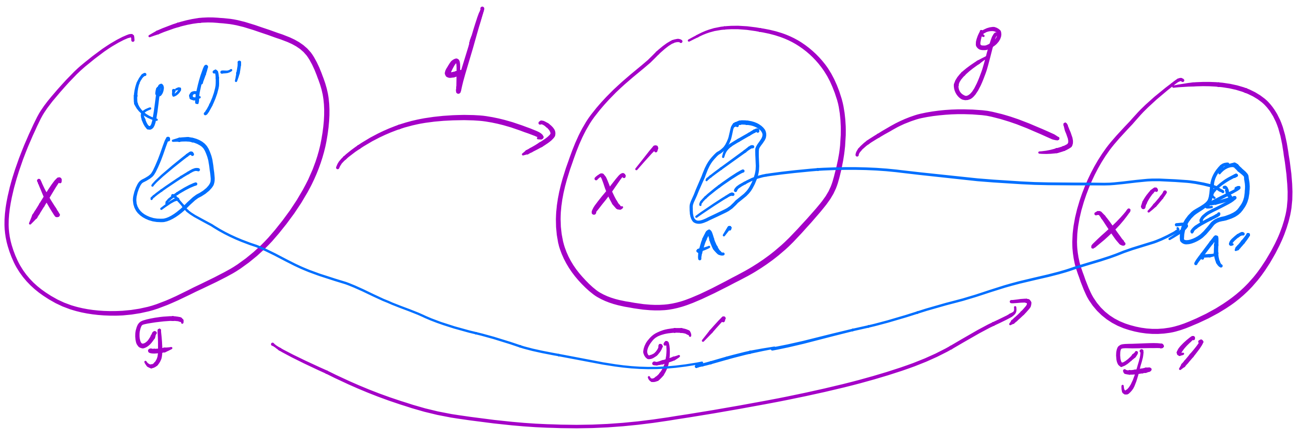
$\blacksquare$

Exercise 6.4. Let $f_k : X \to \mathbb{R}$ be $\mathcal{F}$-measurable functions, $k = 1, \dots, m$. We consider the function
$$f = (f_1, \dots, f_m) : X \to \mathbb{R}^m$$

Show that f is $\mathcal{F}$-measurable, that is, $\forall\ A' \in \mathcal{B}(\mathbb{R}^m)\ f^{-1}(A') \in \mathcal{F}$.

Hint. Take $\mathcal{H} = \{ [a_1, b_1) \times \dots \times [a_m, b_m) : a_k < b_k \}$ and use Th. 6.1.

2. Composition of measurable maps



$$g\circ f = g(f)$$

Th 6.5 Let $(X,\mathcal{F})$, $(X',\mathcal{F}')$, $(X'',\mathcal{F}'')$ be measurable spaces, $f: X \to X'$ be $(\mathcal{F},\mathcal{F}')$-measurable and $g: X' \to X''$ be $(\mathcal{F}',\mathcal{F}'')$-measurable. Then $f\circ g$ is $(\mathcal{F},\mathcal{F}'')$-measurable.

Proof Take $A'' \in \mathcal{F}''$. Then the set

$$A' := g^{-1}(A'') = \{y \in X' : g(y) \in A''\}$$

because g is $(\mathcal{F}',\mathcal{F}'')$-measurable. Consider

$$(g\circ f)^{-1}(A'') = \{x \in X : \underbrace{g(f(x)) \in A''}_{\Leftrightarrow\, f(x)\in A'}\} =$$

$$= \{x \in X : f(x) \in A'\} =$$

$$= f^{-1}(A') \in \mathcal{F}.$$

∎

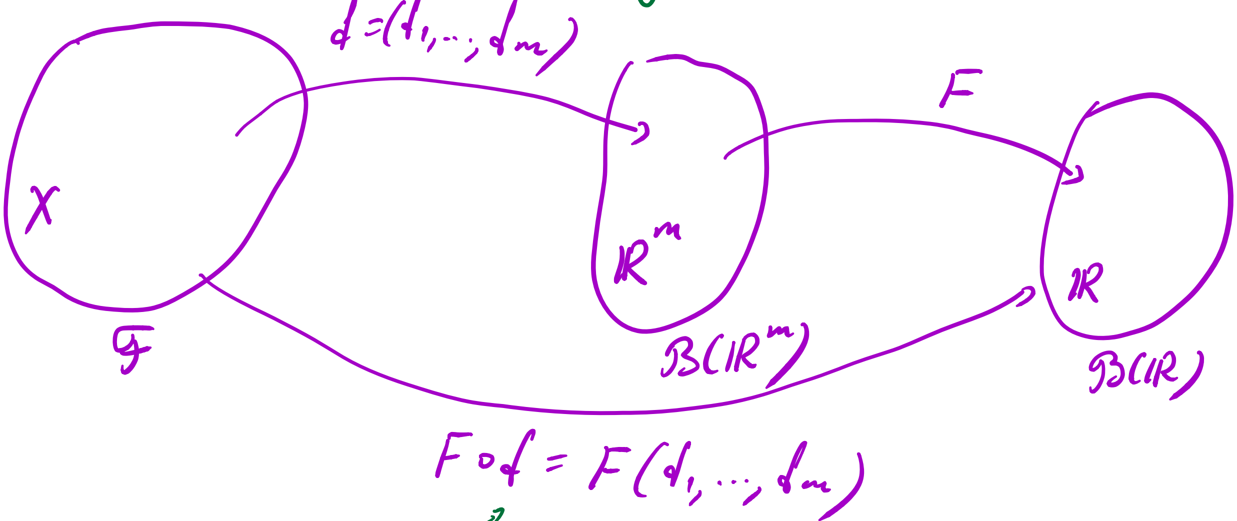
Corollary 6.6 Let $(X, \mathcal{F})$ be a measurable space, $f_k : X \to \mathbb{R}$ be $\mathcal{F}$-measurable, $k = 1, \dots, m$. Let $F : \mathbb{R}^m \to \mathbb{R}$ – Borel measurable. Then

$$F(f_1, \dots, f_m) : X \to \mathbb{R}$$

is $\mathcal{F}$-measurable.

Proof.

$\mathcal{F}$-measurable by Ex. 6.4



$\mathcal{F}$-measurable by Cor. 6.6.

∎

3. Properties of measurable functions

Th 6.7 Let $(X, \mathcal{F})$ be a measurable space and $f_1, f_2 : X \to \mathbb{R}$ be $\mathcal{F}$-measurable functions. Then the following functions: cf_1, $f_1 \pm f_2$, $f_1 \cdot f_2$, $\frac{f_1}{f_2}$ (if $f_2(x) \neq 0$, $x \in X$), $\min\{f_1, f_2\}$,

$\max\{f_1, f_2\}$

are $\mathcal{F}$-measurable.

Proof. The statement follows from Cor. 6.6. Indeed, e.g. $f_1 + f_2$ is $\mathcal{F}$-measurable, since

$$f_1 + f_2 = F(f_1, f_2),$$

where $F(u,v) = u+v$ is Borel measurable as a continuous function.

Th. 6.8 Let $(X, \mathcal{F})$ be a measurable space and $f_n : X \to \mathbb{R}$, $n \geq 1$ be a sequence of $\mathcal{F}$-measurable functions. Then the following functions

1) $g_1(x) := \sup\limits_{n\geq 1} f_n(x)$, 2) $g_2(x) := \inf\limits_{n\geq 1} f_n(x)$

3) $f_3(x) := \overline{\lim\limits_{n\to\infty}} f_n(x)$, 4) $f_4(x) := \underline{\lim\limits_{n\to\infty}} f_n(x)$

are $\mathcal{F}$-measurable.

In particular, the function $f(x) := \lim\limits_{n\to\infty} f_n(x)$, $x \in X$, if the limit exists $\forall x$ is also $\mathcal{F}$ measurable.

The set

$$C := \{x \in X : \{f_n(x)\}_{n\geq 1} \text{ converges in } \mathbb{R}\} \in \mathcal{F}.$$

Proof 1) $\forall a \in \mathbb{R}$

$$g_1^{-1}((-\infty, a]) = \{x : g_1(x) \leq a\} =$$

$$= \{x : \sup_{n\geq 1} f_n(x) \leq a\} =$$

$$= \bigcap_{n=1}^{\infty} \underbrace{\{f_n(x) \leq a\}}_{\in \mathcal{F}} \in \mathcal{F}$$

2) $\forall a \in \mathbb{R}$

$$g_2^{-1}([a, +\infty)) = \{x : g_2(x) \geq a\} =$$

$$= \{x : \inf_{n\geq 1} f_n(x) \geq a\} =$$

$$= \bigcap_{n=1}^{\infty} \underbrace{\{f_n(x) \geq a\}}_{\in \mathcal{F}} \in \mathcal{F}.$$

3) $$g_3(x) = \underbrace{\inf_{n\geq 1} \underbrace{\sup_{k\geq n} f_k(x)}_{\mathcal{F}\text{-measurable by 1)}}}_{\mathcal{F}\text{-measurable by 2)}}$$

4) $$g_4(x) = \sup_{n\geq 1} \inf_{k\geq n} f_k(x)$$

$$C = \{x : g_3(x) = g_4(x)\} = \{x : g_3(x) - g_4(x) = 0\} =$$

$$= (g_3 - g_4)^{-1}(\{0\}) \in \mathcal{F}$$

Because $\{0\} \in \mathcal{B}(\mathbb{R})$.

$\blacksquare$