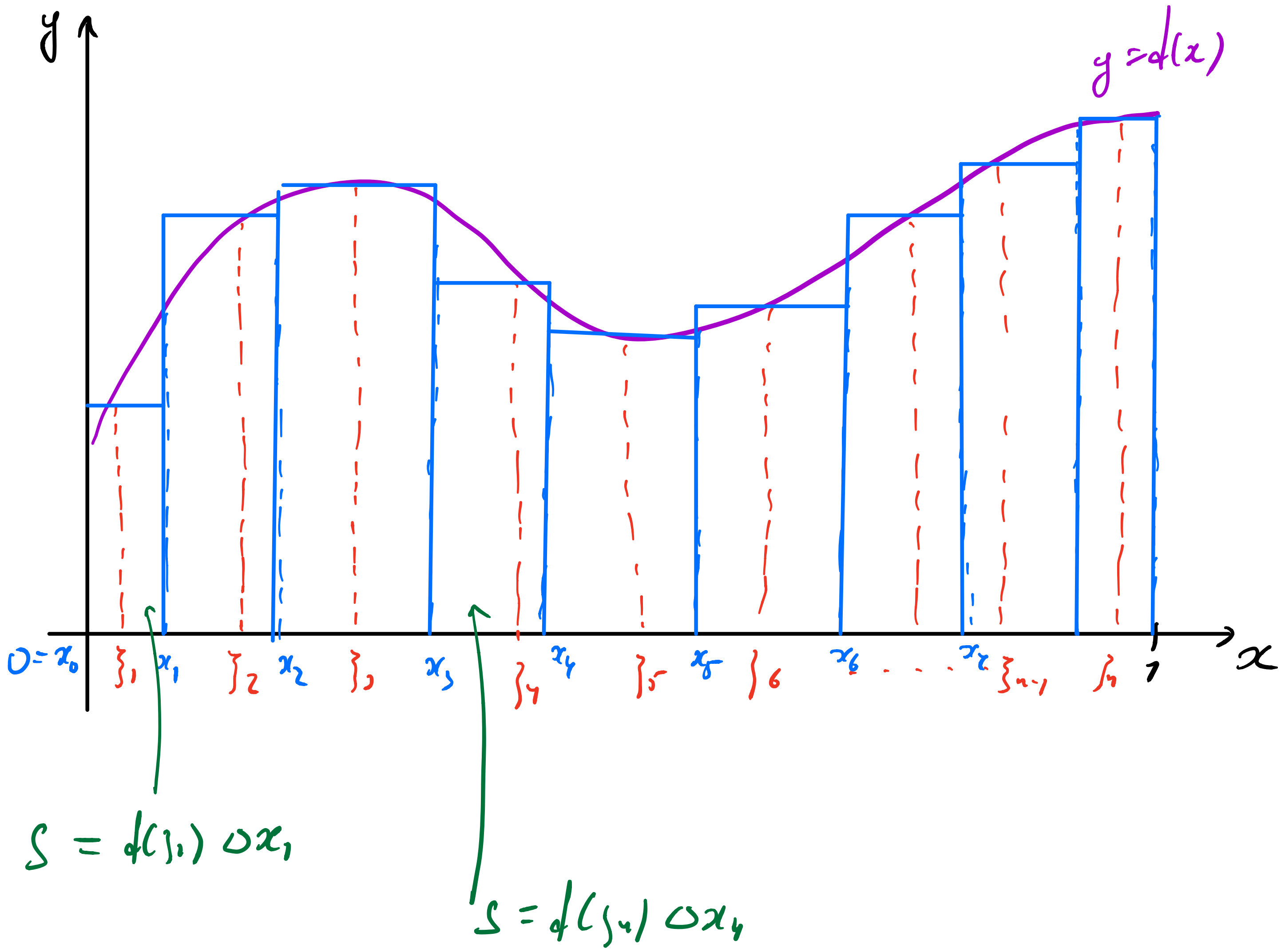
5. Measurable functions.

1. Motivation of definition. Idea of the introduction of the Lebesgue integral.

Let $X = [0,1]$ and $f : [0,1] \to \mathbb{R}$ be a function. Let us recall the definition of the Riemann integral.



We define the Riemann sums

$$S_n = \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k, \qquad x_k = x_k - x_{k-1}$$

We say that f is Riemann integrable if there exists the limit

$$(5.1) \qquad \lim_{|\Delta x|\to 0} \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k \quad , \quad |\Delta x| := \max_k |\Delta x_k|$$

which does not depend on the choise of $\{\xi_k\}$.

Limit (5.1) is called the Riemann integral of f and is denoted by

$$\int_0^1 f(x)\,dx.$$

Example 5.1 a) If f is a continuous function, then f is Riemann integrable.

b) $$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1], \\ 0, & x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

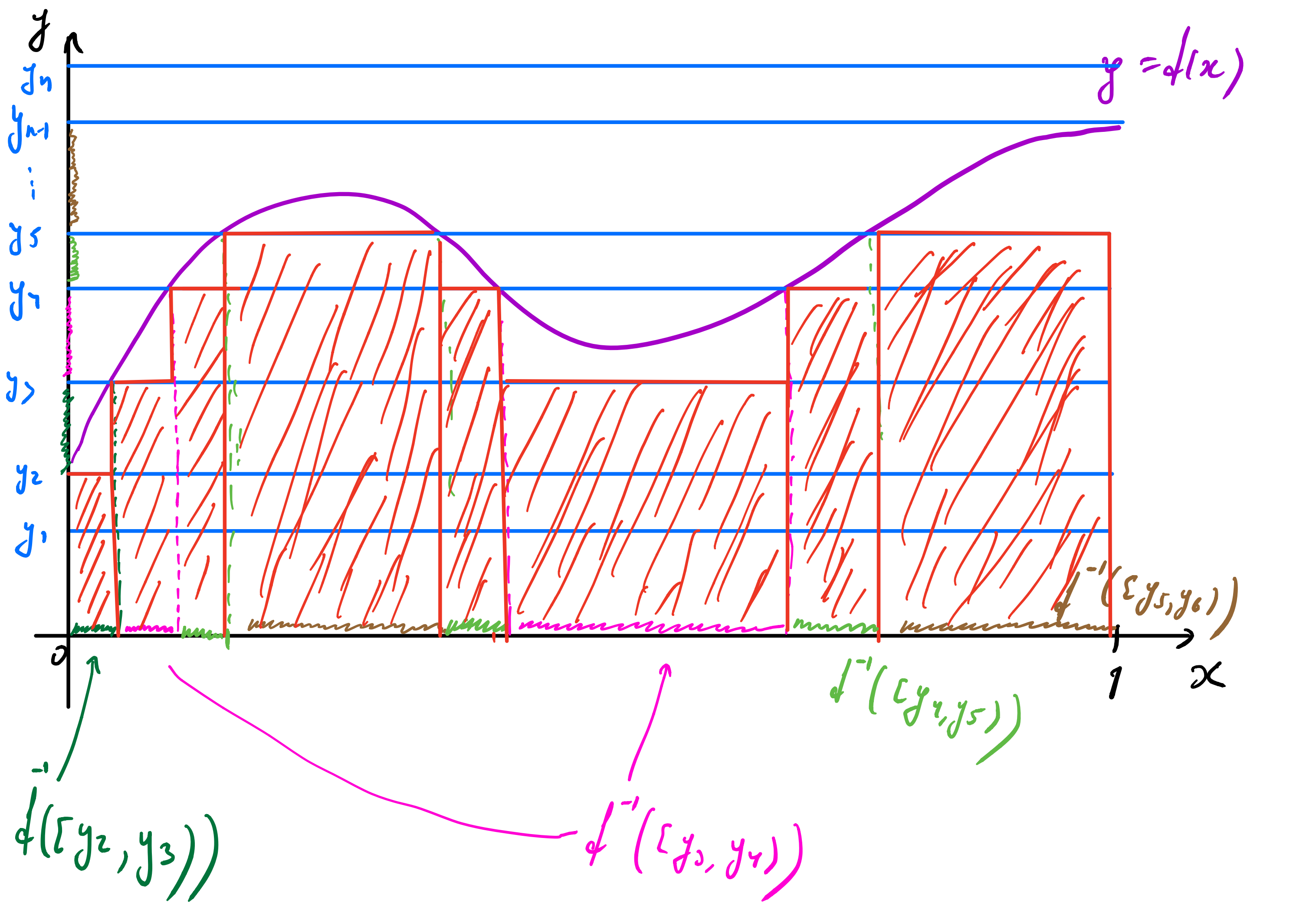
This function is not Riemann integrable since the limit depends on the choise of $\{\xi_k, k=1,\dots,n\}$. Indeed, if $\xi_k \in [x_{k-1}, x_k]$, $k=1,\dots,n$ are rational then

$$\sum_{k=1}^{n} f(\xi_k)\,\Delta x_k = \sum_{k=1}^{n} 1\,\Delta x_k = 1$$

if they are irrational, then

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k = \sum_{k=1}^{n} 0\,\Delta x_k = 0.$$

Other approach to the definition of the integral.

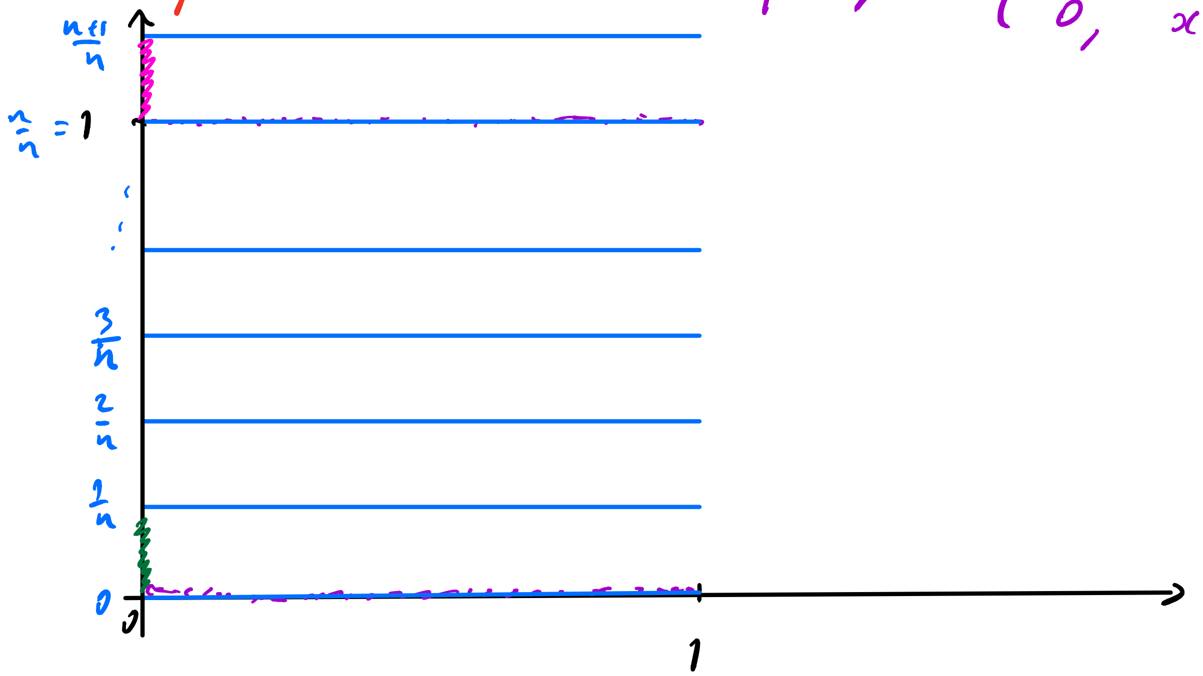


Consequently, we can define the integral as

$$\lim_{|\Delta y| \to 0} \sum_{k=1}^{n} y_k \, \lambda\left(f^{-1}([y_k, y_{k+1}))\right) = \int_a^b f(x)\,\lambda(dx)$$

Remark that $\int_a^b f(x)\,dx = \int_a^b f(x)\,\lambda(dx)$, if f is continuous. But the new definition is better.

Example 5.2 Take $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1], \\ 0, & x \notin [0,1] \setminus \mathbb{Q}. \end{cases}$



$$f^{-1}\left(\left[0, \tfrac{1}{n}\right)\right) = [0,1] \setminus \mathbb{Q}$$

$$f^{-1}\left(\left[\tfrac{n}{n}, \tfrac{n+1}{n}\right)\right) = \mathbb{Q} \cap [0,1]$$

$$f^{-1}\left(\left[\tfrac{k}{n}, \tfrac{k+1}{n}\right)\right) = \emptyset, \quad k \neq 0, n.$$

Hence

$$\sum_{k=0}^{n} \frac{k}{n} \lambda\left(f^{-1}\left(\left[\tfrac{k}{n}, \tfrac{k+1}{n}\right)\right)\right) =$$

$$= 0 \cdot \lambda\Big(\underbrace{[0,1] \setminus \mathbb{Q}}_{=1}\Big) + \frac{1}{n} \lambda(\emptyset) + \cdots$$

$$+ \frac{n-1}{n}\lambda(\emptyset) + \frac{n}{n}\lambda(\underbrace{\mathbb{Q}\cap[0,1]}_{=0}) =$$

$$= 0.$$

Consequently

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} 0 = 0.$$

From this approach of the integral we need to be sure that we can compute the Lebesgue measure of sets

$$A_k = f^{-1}([y_k, y_{k+1})),$$

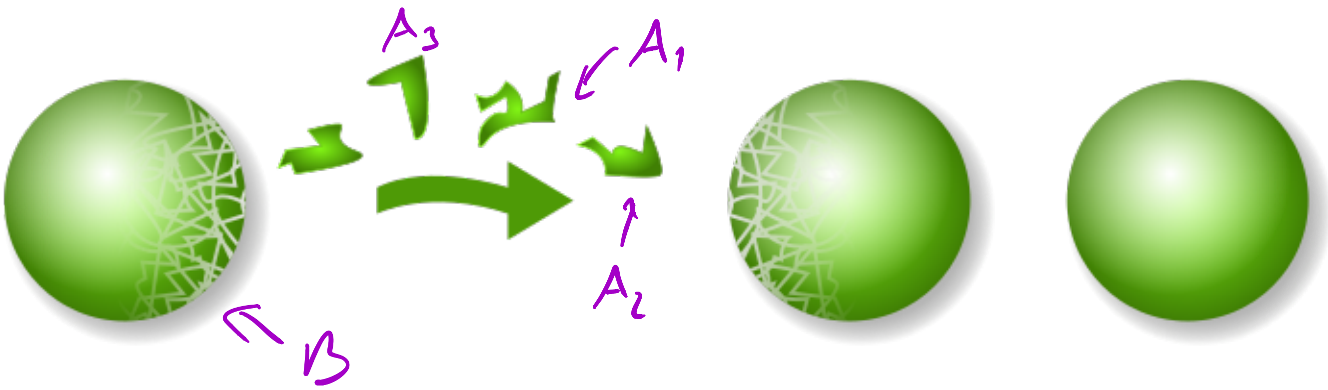
that is, the sets A_k, $k=1,\dots,n$, has to be Lebesgue (or Borel) measurable sets.

Remark 5.3 Not all subsets of $\mathbb{R}^d$ are Lebesgue measurable.

Banach – Tarski "paradox":

Given a solid ball in 3-dimensional

space, there exists a decomposition of the ball into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original ball



The Banach - Tarski "paradox" is a strong mathematical fact. we do not have any contradiction here since the pieces are not Lebesgue measurable:

$$V(B) = \sum_{k=1}^{n} \underbrace{V(A_k)}_{\text{does not exists.}} = 2V(B).$$

So, we have no contradiction here.

2. Measurable functions. Definition.

Let X, X' be some sets, and

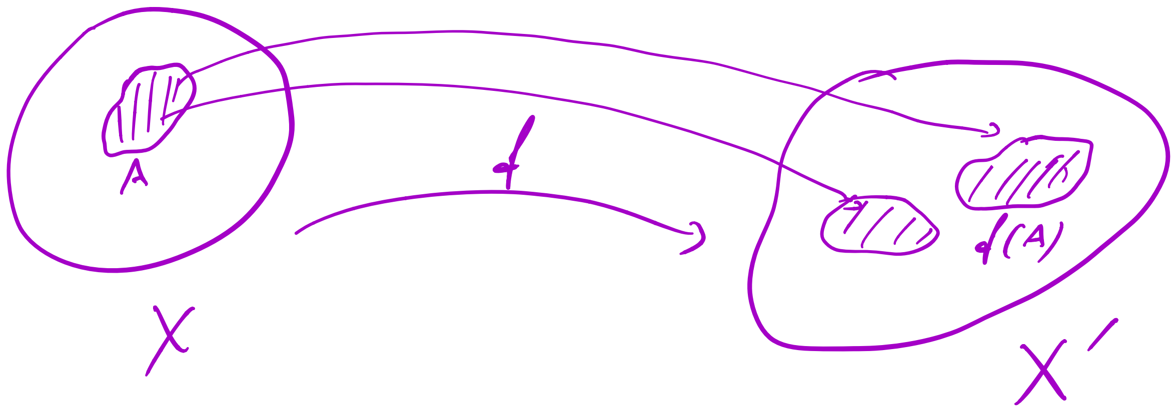
$$f : X \to X'$$

be a map.

Def 5.4. (i) For $A \subset X$ the set

$$f(A) = \{ f(x) : x \in A \}$$

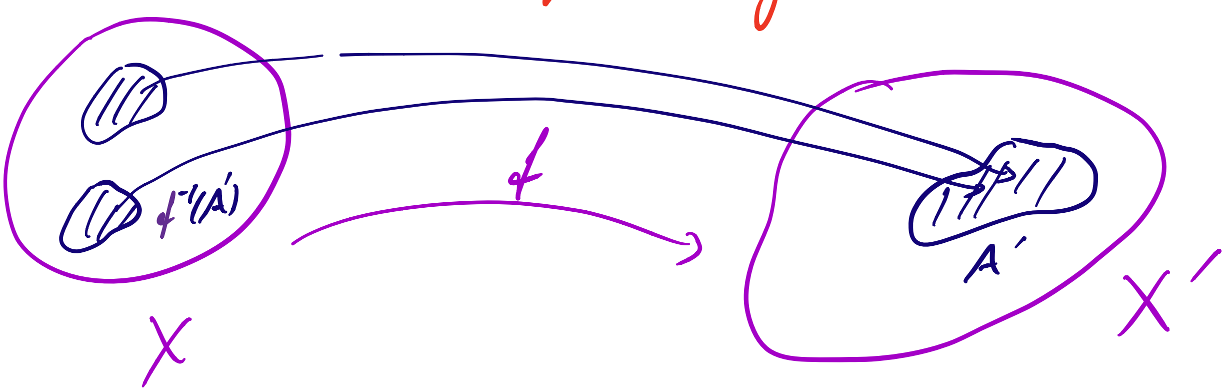
is called the *image of A*



(ii) For a set $A' \subset X'$ the set

$$f^{-1}(A') = \{ x \in X : f(x) \in A' \}$$

is called the *preimage of A'*



Exercise 5.5. Show that

a) $f^{-1}\left(\bigcup_{k=1}^{n} A'_k\right) = \bigcup_{k=1}^{n} f^{-1}(A'_k)$

b) $f^{-1}\left(\bigcap_{k=1}^{n} A'_k\right) = \bigcap_{k=1}^{n} f^{-1}(A'_k)$

c) $f^{-1}(B' \setminus A') = f^{-1}(B') \setminus f^{-1}(A')$,

where $A'_k \subset X'$, $B', A' \subset X$ and $n \in \mathbb{N} \cup \{+\infty\}$.

Solution a)

$$f^{-1}\left(\bigcup_{k=1}^{n} A'_k\right) = \left\{x : f(x) \in \bigcup_{k=1}^{n} A'_k\right\} =$$

$$= \bigcup_{k=1}^{n} \{x : f(x) \in A'_k\} = \bigcup_{k=1}^{n} f^{-1}(A'_k)$$

Def 5.6. If X is a set and $\mathcal{F}$ is a σ-algebra on X, then $(X, \mathcal{F})$ is called a **measurable space**.

Def 5.7. a) Let $(X, \mathcal{F})$ and $(X', \mathcal{F}')$ be measurable spaces and $f : X \to X'$.

The function f is called $(\mathcal{F}, \mathcal{F}')$ measurable if

$$\forall A' \in \mathcal{F}' \qquad f^{-1}(A') \in \mathcal{F}.$$

b) In the case $X' = \mathbb{R}$, $\mathcal{F}' = \mathcal{B}(\mathbb{R})$, then f is called $\mathcal{F}$-measurable

c) If additionally $X = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$, i.e.

$f : \mathbb{R} \to \mathbb{R}$. and

$$\forall A' \in \mathcal{B}(\mathbb{R}) \qquad f^{-1}(A') \in \mathcal{B}(\mathbb{R}),$$

then f is called Borel measurable.

Example 5.8 Let $X = [0,1]$, $\mathcal{F} = \{\emptyset, X\}$
$X' = \mathbb{R}$, $\mathcal{F}' = \mathcal{B}(\mathbb{R})$

Then only constant functions are $\mathcal{F}$-measurable. Indeed, we know that

$A' = \{y\} \in \mathcal{F}' = \mathcal{B}(\mathbb{R})$. So,

$$f^{-1}(A') \in \mathcal{F}$$

means that $f^{-1}(\{y\}) = \{x : f(x) = y\} = \emptyset$
or $[0,1]$

It means, that $f(x) = c$, $\forall x \in [0,1]$, where c is a constant.

Example 5.9 $X = X' = \mathbb{R}$, $\mathcal{F} = \mathcal{F}' = \mathcal{B}(\mathbb{R})$

$f(x) = x$. Then f is Borel measurable, since if $A' \in \mathcal{B}(\mathbb{R})$, then $f^{-1}(A') = A' \in \mathcal{B}(\mathbb{R})$.

Remark 5.10 The definitoon of measurability is very similar to one of definitions of continuity. Indeed, f is continuous if and only if the preimage of every open set is an open set.
(For the measurability we require that the preimage of any measurable set is measurable)