4. Measures: extension

1. Extension of a measure from semi-ring to the generated ring.

Let X be a universal set and $H \subseteq 2^X$. We recal that a nonnegative and σ-additive function μ defined on a semiring H is called a **measure**. That is, a measure

$$\mu : H \to \mathbb{R}$$

has to satisfy the following properties

1) $\forall A \in H \quad \mu(A) \geq 0$

2) $\forall A_n \in H,\ n \geq 1,\ A_n \cap A_m = \emptyset$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Today, we will discuss a possibility of extension of a measure from a semiring to a σ-ring. In this section, we consider the extension to a ring.

We will assume that $\mathcal{H}$ is a semiring. We also recall that Theorem 2.11 implies that

$$\tau(\mathcal{H}) = \left\{ \bigcup_{k=1}^{n} A_k : A_1, \dots, A_n \in \mathcal{H}, n \geq 1 \right\}$$

Example 4.1 $\mathcal{H} = \{ [a,b) : a < b \} \cup \{\emptyset\}$

$$\tau(\mathcal{H}) = \left\{ \bigcup_{k=1}^{n} [a_k, b_k) : a_k < b_k, n \geq 1 \right\} \cup \{\emptyset\}$$

Say, the set

$$[2,5) \cup [7,10) \cup [9,11) =$$

$$= [2,5) \cup [7,11) \in \tau(\mathcal{H}).$$

Theorem 4.2 Let μ be a measure on a semiring $\mathcal{H}$. The measure μ can be extended to a measure on $\tau(\mathcal{H})$ and this extension is unique. Moreover, if a measure μ is finite (take finite values), then the extension is finite.

This extension is given as follows

Let $A \in \tau(\mathcal{H})$, i.e. $A = \bigcup_{k=1}^{n} A_k$, $A_k \in \mathcal{H}$.

We first remark that there exists $C_1, C_2, \dots, C_m \in \mathcal{H}$ s.t. $C_k \cap C_j = \emptyset$ and

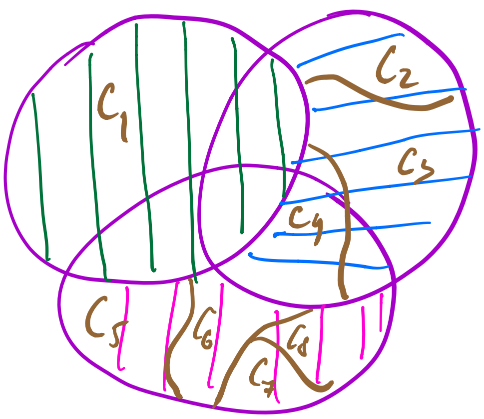
$$A = \bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{m} C_k \qquad (4.1)$$

Then

$$\mu(A) := \sum_{k=1}^{m} \mu(C_k).$$

Let us explain (4.1):

$$A = \bigcup_{k=1}^{3} A_k = A_1 \cup \overbrace{(A_2 \setminus A_1)}^{\text{can be written as a union of nonintersecting sets from } \mathcal{H} \text{ by def. of semiring}} \cup \big(\underbrace{A_3 \setminus (A_1 \cup A_2)}_{\text{can also be written as ....}}\big)$$



$$= C_1 \cup \dots \cup C_8 .$$

2. Outer measure

Def 4.3 A function $\lambda^* : 2^X \to (-\infty, +\infty]$ is called an *outer measure* if:

(i) $\lambda^*(\emptyset) = 0$, λ^* is nonnegative

(ii) $\forall A, A_n \in 2^X$, $A \subset \bigcup_{n=1}^{\infty} A_n$

$$\lambda^*(A) \le \sum_{n=1}^{\infty} \lambda^*(A_n).$$ (σ-semiadditivity)

Def 4.4 Let μ be a measure on a ring $R \subseteq 2^X$. For any $A \in 2^X$ set

$$\mu^*(A) = \begin{cases} 0, & A = \emptyset \\ \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : A_n \in R,\ n \ge 1,\ A \subseteq \bigcup_{n=1}^{\infty} A_n\right\}, & \text{otherwise.} \end{cases}$$

Theorem 4.5 μ^* is an outer measure on R.

Proof We only need to show that for any A, $A_n \in 2^X$, $n \ge 1$, $A \subseteq \bigcup_{n=1}^{\infty} A_n$ we have

$$\mu^*(A) \le \sum_{n=1}^{\infty} \mu^*(A_n).$$

It is enough to show this only in the case $\mu^*(A_n) < +\infty$, $n \ge 1$.

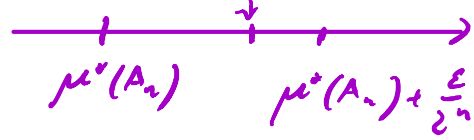
Let $\varepsilon > 0$. According to Def 4.4.

$\forall A_n \quad \exists \quad B_k^n \in R,\ k \ge 1$ s.t.

$$A_n \subseteq \bigcup_{k=1}^{\infty} B_{k,n}$$

and

$$\sum_{k=1}^{\infty} \mu(B_{k,n}) < \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$



Hence,

$$A \subset \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} B_{k,n}$$

By Def. 4.4.

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(B_{k,n}) < \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) =$$

$$= \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

Making $\varepsilon \to 0+$, we have

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

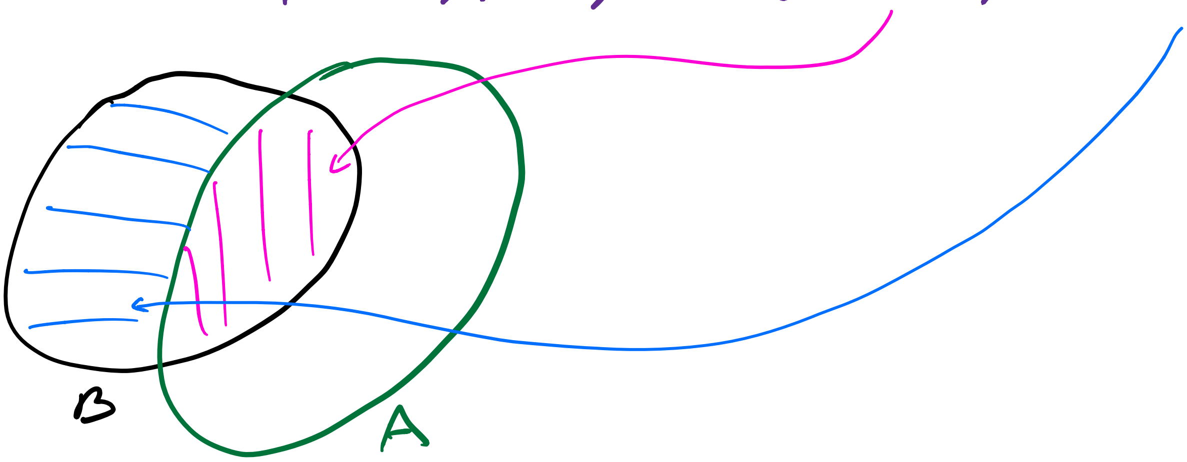
∎

Def. 4.6. The function μ^* from Def. 4.4. is called the outer measure generated by the measure μ.

3. λ^* - measurable sets. Carathéodory theorem.

Def 4.7. Let λ^* be an outer measure on 2^X. A set A is called λ^*-measurable, if

$$\forall\, B \subseteq X \quad \lambda^*(B) = \lambda^*(B \cap A) + \lambda^*(B \setminus A) \quad (4.2)$$



Remark 4.8 By the definition of outer measure the inequality

$$\lambda^*(B) \le \lambda^*(B \cap A) + \lambda^*(B \setminus A)$$

is always true, since

$$B \subseteq (B \cap A) \cup (B \setminus A).$$

Theorem 4.9 (Carathéodory) Let λ^* be an outer measure on 2^X and S be a class of all λ^*-measurable sets. Then the class S is a σ-algebra and λ^* is a measure on S (that is, λ^* is σ-additive on S)

Def 4.10. A measure μ on a σ-algebra H is called *complete* if

$$\forall A \in H \text{ s.t. } \mu(A) = 0$$

we have that any subset $C \subseteq A$ also belongs to H.
(In this case, $\mu(C) = 0$ by monotonicity)

Proposition 4.11. Under the assumptions of Theorem 4.9, the measure λ^* is complete on S.

Proof Let $A \in S$ and $\lambda^*(A) = 0$, $C \subseteq A$.
We need to show that $C \in S$.
We need to check (4.2) for C.

Let $B \in 2^X$. By the monotonicity of λ^* (it directly follows from the definition of λ^*)

$$\lambda^*(B) \geq \lambda^*(B \cap C^c) \geq \lambda^*(B \cap A^c) =$$

$$= \underbrace{\lambda^*(B \cap A)}_{=0} + \lambda^*(B \cap \bar{A}) = \lambda^*(B),$$

since $0 \leq \lambda^*(B \cap A) \leq \lambda^*(A) = 0$.

Similarly, $0 \le \lambda^*(B \cap C) \le \lambda^*(A) = 0$

So, $\lambda^*(B) = \lambda^*(B \cap C^c) = \lambda^*(B \setminus C) + \underbrace{\lambda^*(B \cap C)}_{=0}$.

4. μ^*-measurability of sets from the ring.

Let R be a ring and μ be a measure on R. Let also μ^* be the outer measure generated by μ. Let S be a class of all μ^*-measurable subsets od X. We also denote

$$\bar{\mu}(A) = \mu^*(A), \quad A \in S.$$

By the Caratheodory theorem S is a σ-algebra and $\bar{\mu}$ is a measure on S.

Theorem 4.12 $R \subseteq S$ and the measure $\bar{\mu}$ is the extension of μ to S, that is, $\bar{\mu}(A) = \mu(A)$ $\forall A \in R$.

Proof I. We first show that $\forall A \in R$

$$\mu^*(A) = \mu(A).$$

Since $A \subset A \cup \emptyset \cup \emptyset \cup \ldots = \bigcup_{k=1}^{n} A_k$

$$\mu^*(A) \le \sum_{k=1}^{\infty} \mu(A_k) = \mu(A).$$

Now, let $A \subseteq \bigcup_{n=1}^{\infty} A_n$, $A_n \in R$, $n \geq 1$.

Then

$$\underbrace{A}_{\in R} = \bigcup_{n=1}^{\infty} \underbrace{(A \cap A_n)}_{\in R} .$$

By the monotonicity of μ and the σ-semiadditivity

$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A \cap A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Hence, $\underline{\mu(A) \leq \mu^*(A).}$

Consequently, $\mu(A) = \mu^*(A)$.

II. We show that $R \subseteq S$. Let $A \in R$, and $\varepsilon > 0$. We consider any set $B \subset X$, $\mu^*(B) < +\infty$ and show that

$$\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \setminus A).$$

According to Def. 4.4.

$$\exists\, A_n \in R,\ n \geq 1 \text{ s.t. } B \subseteq \bigcup_{n=1}^{\infty} A_n$$

$$\mu^*(B) + \varepsilon > \sum_{n=1}^{\infty} \mu(A_n)$$

So, $\mu^*(B)+\varepsilon > \sum_{n=1}^{\infty}\mu(A_n) \overset{\text{additivity of } \mu}{=}$

$$= \sum_{n=1}^{\infty}\left(\mu(A_n\cap A)+\mu(A_n\setminus A)\right) \overset{\text{Def. 4.4}}{\geq}$$

$$\geq \mu^*(B\cap A)+\mu^*(B\setminus A),$$

since $B\cap A \subseteq \bigcup_{n=1}^{\infty} A_n\cap A$, $B\setminus A \subseteq \bigcup_{n=1}^{\infty} A_n\setminus A$.

This and Remark 4.8 implies that

$$\mu^*(B) = \mu^*(B\cap A)+\mu^*(B\setminus A).$$

$\blacksquare$

4. Lebesgue measure.

Let $X=\mathbb{R}$, $H=\{(a,b],\ a<b\}\cup\{\emptyset\}$ (H is a semiring). Define

$$\lambda(\phi)=0;\qquad \lambda((a,b]) := b-a,\qquad a<b$$

Then, by Corollary 3.13, λ is a measure on H. Next, by Th. 4.2 there exists an extension of λ to the ring $r(H)$ generated by H.

Next, let S be the class of all λ^*-measurable subsets of $X=\mathbb{R}$.

The Caratheodory theorem implies that S is a σ-algebra and λ^* is a measure on S. Moreover, Th. 4.12 implies that

$$H \subset \tau(H) \subset S$$

Since, $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra which contains all sets from H, we have $\mathcal{B}(\mathbb{R}) \subset S$.

Hence,
$$H \subset \tau(H) \subset \mathcal{B}(\mathbb{R}) \subset S.$$

We also remark that λ^* is the extension of λ from $\tau(H)$ to S by Th 4.13.

Def 4.13 • Sets from S are called Lebesgue measurable sets.

• The measure λ^* defined on S is called the Lebesgue measure on $\mathbb{R}$.

Remark 4.14. The extension of λ to $\mathcal{B}(\mathbb{R})$ is unique.

Remark 4.15 We can define the Lebesgue measure on $\mathbb{R}^d$, taking

$$\mathcal{H} = \{ (a_1, b_1] \times \dots \times (a_n, b_n] : a_k < b_k, k=1,\dots,n\} \cup \{\emptyset\}$$

and

$$\lambda(\emptyset) = 0; \quad \lambda((a_1,b_1] \times \dots \times (a_n, b_n]) = $$
$$= (b_1 - a_1) \cdot \dots \cdot (b_n - a_n).$$

Example 4.16 a) Let $x \in \mathbb{R}$. Then

$$\{x\} = \bigcap_{n=1}^{\infty} (x - \tfrac{1}{n}, x]$$

By Th. 3.10

$$\lambda(\{x\}) = \lim_{n \to \infty} \lambda((x - \tfrac{1}{n}, x]) =$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0.$$

b) $$\lambda(\mathbb{Q}) = \lambda\left(\bigcup_{n=1}^{\infty} \{r_n\}\right) =$$

$$= \sum_{n=1}^{\infty} \lambda(\{r_n\}) = \sum_{n=0}^{\infty} 0 = 0,$$

where $\mathbb{Q} = \{r_1, r_2, r_3, \dots \}$ is the set of rational numbers.