3. Measures: properties

1. Definition of a measure and basic properties.

Let X be a fundamental set and $H \subset 2^X$ is a class of sets.
the main object of the measure theory are functions

$$\mu : H \to (-\infty, \infty),$$

which satisfies a spetial requirements.
Length, area, volume are real examples of such functions. They lead to a class of functions which satisfy spetial properties. For example, the area is non-negative; the area of two nonintersecting sets equals sum of areas of those sets.
We will transfer this spetial properties on an abstract situation.

We will assume that μ can take the value $+\infty$. Moreover, we assume that

$$(+\infty) + (+\infty) = +\infty$$

$\forall a \in \mathbb{R}$ $\quad a < +\infty$, $\quad a + \infty = \infty$.

We start from the following definition

Def 3.1 A function $\mu : \mathcal{H} \to (-\infty, +\infty]$ is called

(i) nonnegative if $\mu(A) \geq 0 \quad \forall A \in \mathcal{H}$

(ii) countably additive (or σ-additive) if $\forall A_n \in \mathcal{H}$, $n \geq 1$, $A_k \cap A_n = \emptyset$, $k \neq n$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Def 3.2 A measure is a nonnegative and σ-additive function on a semiring

Remark 3.3 If μ is a measure on $\mathcal{H}$, then $\mu(\emptyset) = 0$. Indeed, take $A \in \mathcal{H}$ with $\mu(A) < \infty$

$A_1 = A$, $\quad A_2 = A_3 = A_4 = \ldots = \emptyset \in \mathcal{H}$

(check that $\emptyset \in \mathcal{H}$ if $\mathcal{H}$ is a semiring)

Then
$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) =$$
$$= \sum_{n=2}^{\infty} \mu(\emptyset) + \mu(A)$$

$$\Rightarrow \mu(\emptyset) = 0.$$

Remark 3.4 A measure is also an additive function, i.e.

$$\forall A_k \in \mathcal{M},\ k=1,\dots,n,\quad A_k \cap A_j = \emptyset,\ k \neq j$$

$$\mu\Big(\bigcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \mu(A_k).$$

This follows from Remark 3.3 because we can take $A_{n+1} = A_{n+2} = \dots \emptyset$. Then

$$\mu\Big(\bigcup_{k=1}^{n} A_k\Big) = \mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mu(A_k) =$$

$$= \sum_{k=1}^{n} \mu(A_k) + \mu(\underbrace{A_{n+1}}_{=\emptyset}) + \dots = \sum_{k=1}^{n} \mu(A_k).$$

Example 3.5 Let $X = \mathbb{N} = \{1, 2, 3, \dots\}$

$\mathcal{M} = 2^X$.

We set $\mu(A) = \begin{cases} \text{number of elements of } A, & \text{if } A \text{ finite} \\ +\infty, & \text{if } A \text{ infinite}. \end{cases}$

e.g. $\mu(\{1, 7, 8, 10\}) = 4$

$\mu(\{\text{even numbers}\}) = +\infty.$

It is easy to see that μ is a measure.

Exercise 3.6 Let $X = \{x_1, x_2, \dots, x_n, \dots\}$, $M = 2^X$. Take numbers $p_n \geq 0$, $n \geq 1$ such that $\sum_{n=1}^{\infty} p_n = 1$, and set

$$\mu(A) = \sum_{n : x_n \in A} p_n , \quad A \in M.$$

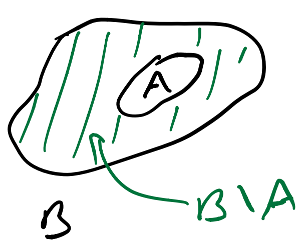
(e.g. $\mu(\{x_1, x_{10}, x_{100}\}) = p_1 + p_{10} + p_{100}$.)

Prove that μ is a measure on M.

Theorem 3.7 Let R be a ring and μ be a measure on R. Then

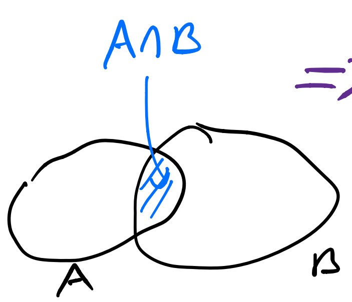
1) μ is monotone on R, that is $\forall A, B \in R$ such that $A \subset B \Rightarrow$
$$\mu(A) \leq \mu(B)$$

2) $\forall A, B \in R$ s.t. $A \subset B$, $\mu(A) < +\infty \Rightarrow$



$$\mu(B \setminus A) = \mu(B) - \mu(A)$$

3) $\forall A, B \in R$ s.t. $\mu(A) < \infty$ or $\mu(B) < \infty$



$$\Rightarrow \mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

4) $\forall B_1, \dots, B_n, A \in R$ s.t. $A \subseteq \bigcup_{k=1}^{n} B_k \Rightarrow$

$$\mu(A) \le \sum_{k=1}^{n} \mu(B_k)$$

5) μ is σ- semiadditive, that is,

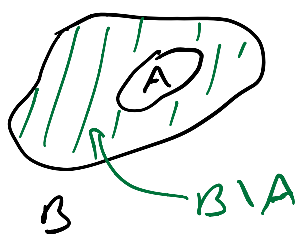
$\forall A_1, A_2, \dots \in R$ s.t. $\bigcup_{n=1}^{\infty} A_n \in R \Rightarrow$

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \le \sum_{n=1}^{\infty} \mu(A_n)$$

(note that here we do not assume that $A_n \cap A_k = \emptyset$, $n \ne k$)

Proof 1) Let $A, B \in R$ s.t. $A \subset B$.

Then
$$B = A \cup (B \setminus A)$$



and
$$A \cap (B \setminus A) = \emptyset.$$

By Remark 3.4

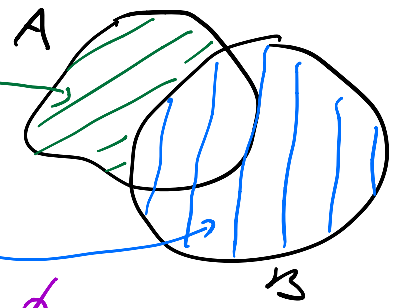
(3.1) $$\mu(B) = \mu(A) + \underbrace{\mu(B \setminus A)}_{\ge 0} \ge \mu(A)$$

2) If $\mu(A) < \infty$. Then (3.1) implies

$$\mu(B \setminus A) = \mu(B) - \mu(A).$$

3) if $\mu(A) < \infty$ or $\mu(B) < \infty$, then $\mu(A \cap B) < \infty$, by 1).

We can write



$$A \cup B = \big(A \setminus (A \cap B)\big) \cup B,$$

$$\big(A \setminus (A \cap B)\big) \cap B = \emptyset$$

Then

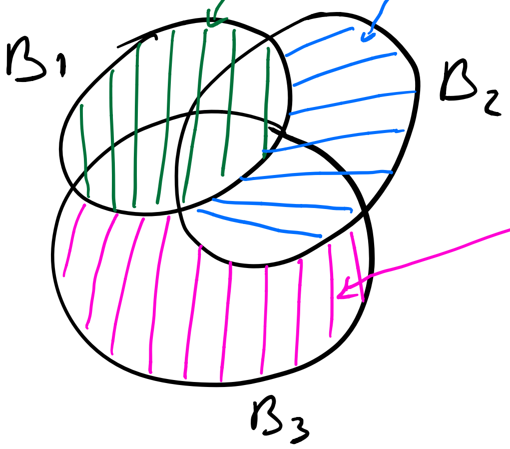
$$\mu(A \cup B) \overset{\text{Remark 3.4}}{=} \mu\big(A \setminus (A \cap B)\big) + \mu(B) =$$

$$\overset{2)}{=} \mu(A) - \mu(A \cap B) + \mu(B).$$

4) Remark that

$$\bigcup_{k=1}^{n} B_k = B_1 \cup (B_2 \setminus B_1) \cup \big(B_3 \setminus (B_1 \cup B_2)\big) \cup \dots \cup \Big(B_n \setminus \bigcup_{k=1}^{n-1} B_k\Big)$$

$n = 3$



Then $\mu(A) \overset{1)}{\leq} \mu\Big(\bigcup_{k=1}^{n} B_k\Big) \overset{\text{Rem. 3.4}}{=}$

$$= \sum_{k=1}^{n} \mu\Big(B_k \setminus \Big(\bigcup_{l=1}^{k-1} B_l\Big)\Big) \overset{1)}{\leq}$$

$$\leq \sum_{k=1}^{n} \mu(B_k).$$

5) $$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty}\left(A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right)\right)\right) \overset{\sigma\text{-additivity}}{=}$$

$$= \sum_{n=1}^{\infty} \mu\left(A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right)\right) \overset{1)}{\leq}$$

$$\leq \sum_{n=1}^{\infty} \mu(A_n).$$

$\blacksquare$

Exercise 3.8 Let μ be a measure on a σ-ring H. Let $A_n \in H$ and $\mu(A_n) = 0$, $n \geq 1$. Show that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0.$$

2 Continuity of a measure

Theorem 3.9 (Continuity from below) Let R be a ring and μ be a measure on R. Then for any increasing sequence $A_n \in R$, $n \geq 1$ ($A_n \subseteq A_{n+1}$, $\forall n \geq 1$) such that $\bigcup_{n=1}^{\infty} A_n \in R$ one has

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

Proof Case I. if $\exists n_0$ s.t. $\mu(A_{n_0}) = +\infty$, then $\forall n \geq n_0$ $\mu(A_n) \geq \mu(A_{n_0}) = +\infty$

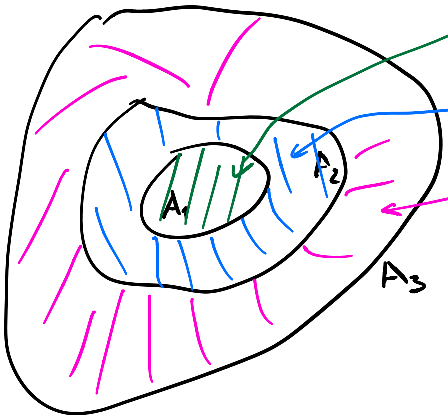
Th. 3.7 1)

and $\mu(\bigcup_{n=1}^{\infty} A_n) \geq \mu(A_{n_0}) = +\infty$

Hence $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n) = +\infty$.

Case II. $\forall n \geq 1$ $\mu(A_n) < +\infty$. Then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \mu(A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \ldots \cup (A_k \setminus A_{k-1}) \cup \ldots)$$



$$= \mu(A_1) + \sum_{k=1}^{\infty} \mu(A_k \setminus A_{k-1})$$

$$= \mu(A_1) + \lim_{n\to\infty} \sum_{k=1}^{n} \mu(A_k \setminus A_{k-1})$$

$$\overset{\text{Th 3.7 2)}}{=} \cancel{\mu(A_1)} + \lim_{n\to\infty} \big(\cancel{\mu(A_2)} - \cancel{\mu(A_1)} + \cancel{\mu(A_3)} - \cancel{\mu(A_2)} + \ldots \mu(A_n) - \cancel{\mu(A_{n-1})} \big)$$

$$= \lim_{n\to\infty} \mu(A_n).$$

∎

Theorem 3.10 Let R be a ring and μ be a measure on R. Then for any decreasing sequence $A_n \in R$, $n \geq 1$ ($A_n \supseteq A_{n+1}$, $n \geq 1$) such that $\bigcap_{n=1}^{\infty} A_n \in R$, and $\mu(A_1) < \infty$

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

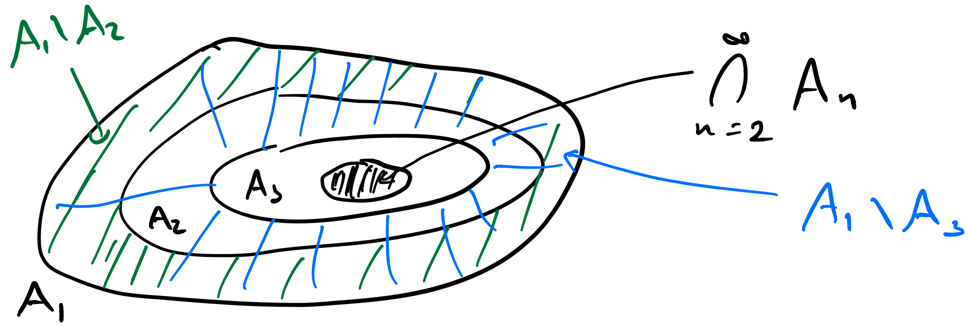
Remark 3.11 The condition $\mu(A_1) < \infty$ is important in Th. 3.10. Indeed, consider the measure from Example 3.5.

Let
$$A_n = \{n, n+1, \dots\}, \quad n \geq 1.$$
Obviously $A_n \supseteq A_{n+1}$, $n \geq 1$. and
$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

So $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$. But $\lim_{n\to\infty} \overbrace{\mu(A_n)}^{=+\infty} = +\infty$.

Proof of Theorem 3.10

$$\mu\left(A_1 \setminus \left(\bigcap_{n=2}^{\infty} A_n\right)\right) = \mu\left(\bigcup_{n=2}^{\infty} \underbrace{(A_1 \setminus A_n)}_{\text{increases}}\right) \overset{Th\ 3.9}{=}$$



$$= \lim_{n\to\infty} \mu(A_1 \setminus A_n) \overset{\text{Th3.7 2)}}{=} \lim_{n\to\infty} (\mu(A_1) - \mu(A_n)).$$

Hence

$$\mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu\left(A_1 \setminus \left(\bigcap_{n=2}^{\infty} A_n\right)\right) =$$

$$= \lim_{n\to\infty} \left(\mu(A_1) - \mu(A_n)\right) \qquad \blacksquare$$

3. Examples of measures

Theorem 3.12 Let R be a ring of all Jordan measurable sets on $\mathbb{R}^d$ and μ be the Jordan measure on R. Then the function μ is σ-additive on R (i.e. it is a measure according to Def. 3.2)

See proof of Theorem ... from [Doropovtsev]

Corollary 3.13 Let $X = \mathbb{R}$ and

$$H = \{(a, b] : -\infty < a < b < +\infty\} \cup \{\emptyset\}$$

(H is a semiring). Then the function

$$\lambda((a, b]) = b - a, \quad \lambda(\emptyset) = 0.$$

is a measure on X.

Theorem 3.14 Let $X = \mathbb{R}$ and

$$H = \{(a, b] : -\infty < a < b < +\infty\} \cup \{\emptyset\}.$$

Let also $F : \mathbb{R} \to \mathbb{R}$ be a non-negative right continuous function on $\mathbb{R}$. Define

$$\lambda_F((a, b]) = F(b) - F(a), \quad a < b$$

$$\lambda_F(\emptyset) = 0.$$

Then the function is a measure on a semiring H.