# 2. Generated classes of sets
# Borel $\sigma$-algebra

## 1. $\sigma$-ring and $\sigma$-algebra

Let $X$ be a fixed set and $2^X$ denote a class of all subsets of $X$.

We recall that $\mathcal{H} \subseteq 2^X$ is

1) semiring if for all $A, B \in \mathcal{H}$

    a) $A \cap B \in \mathcal{H}$

    b) $A \setminus B = \bigcup_{k=1}^{n} C_k$, where $C_k \cap C_j = \emptyset$ $k \neq j$
    $C_k \in \mathcal{H}$, $k = 1, \dots, n$.

2) semialgebra if it is semiring and $X \in \mathcal{H}$

3) ring if $\forall A, B \in \mathcal{H}$

    a) $A \cup B \in \mathcal{H}$

    b) $A \setminus B \in \mathcal{H}$

( ring is a class which is closed with respect to finite number of operations: $\cap, \cup, \setminus$ )

4) algebra if it is ring and $X \in \mathcal{H}$.

( algebra is also closed with respect

to the complement : $A^c = X \setminus A$)

**Def 2.1** • A non-empty class of sets $\mathcal{H} \subset 2^X$ is called a $\sigma$-ring if

(i) $A_1, A_2, \ldots \in \mathcal{H} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{H}$

(ii) $A, B \in \mathcal{H} \Rightarrow A \setminus B \in \mathcal{H}$.

• A class $\mathcal{H}$ is called a $\sigma$-algebra if $A$ is a $\sigma$-ring and $X \in \mathcal{H}$.

**Proposition 2.2** A non-empty class $\mathcal{H}$ is a $\sigma$-algebra if and only if

1) $X \in \mathcal{H}$;

2) $A_1, A_2, \ldots \in \mathcal{H} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{H}$;

3) $A \in \mathcal{H} \Rightarrow A^c \in \mathcal{H}$.

**Proof** The proof is similar to the proof of Proposition 1.11.

**Example 2.3** (Example of ring)

Let $X = \mathbb{R}^2$.

$$\mathcal{H} = \{ A \subseteq \mathbb{R}^d : A \text{ is Jordan measurable } \mu(A) < \infty \}$$

We know that if $A, B \in \mathcal{M}$, that is, $A, B$ are Jordan measurable, then $A \cup B$, $A \setminus B$ are also Jordan measurable and $\mu(A \cup B) < \infty$, $\mu(A \setminus B) < \infty$.
Hence $A \cup B, A \setminus B \in \mathcal{M}$. So, $\mathcal{M}$ is a ring

Note that $\mathcal{M}$ is not a $\sigma$-ring
indeed,

$$\mathbb{Q}^2 = \bigcup_{k=1}^{\infty} A_k, \quad \text{where } A_k = \{r_k\}$$

are Jordan measurable with $\mu(A_k) = 0$.

But $\mathbb{Q}^2$ is not Jordan measurable.

$\mathcal{M}$ is not an algebra (and not $\sigma$-algebra) since $\mathbb{R}^2 \notin \mathcal{M}$ $(\mu(\mathbb{R}^2) \not< \infty)$

Example 2.4

$$X = [0,1]^2$$

$$\mathcal{M} = \{A \subseteq [0,1]^2 : A \text{ is Jordan measurable}\}$$

Then $\mathcal{M}$ is an algebra (now $X \in \mathcal{M}$) but not $\sigma$-algebra.

**Exercise 2.5** Let $H$ be a $\sigma$-ring. Prove that $A_1, A_2, \ldots \in H$ implies that

$$\bigcap_{n=1}^{\infty} A_n \in H.$$

**Remark 2.6** • $\sigma$-ring is a class closed with respect to countable number of operations: $\cap, \cup, \setminus$

• $\sigma$-algebra is additionally closed with respect to taking the complement

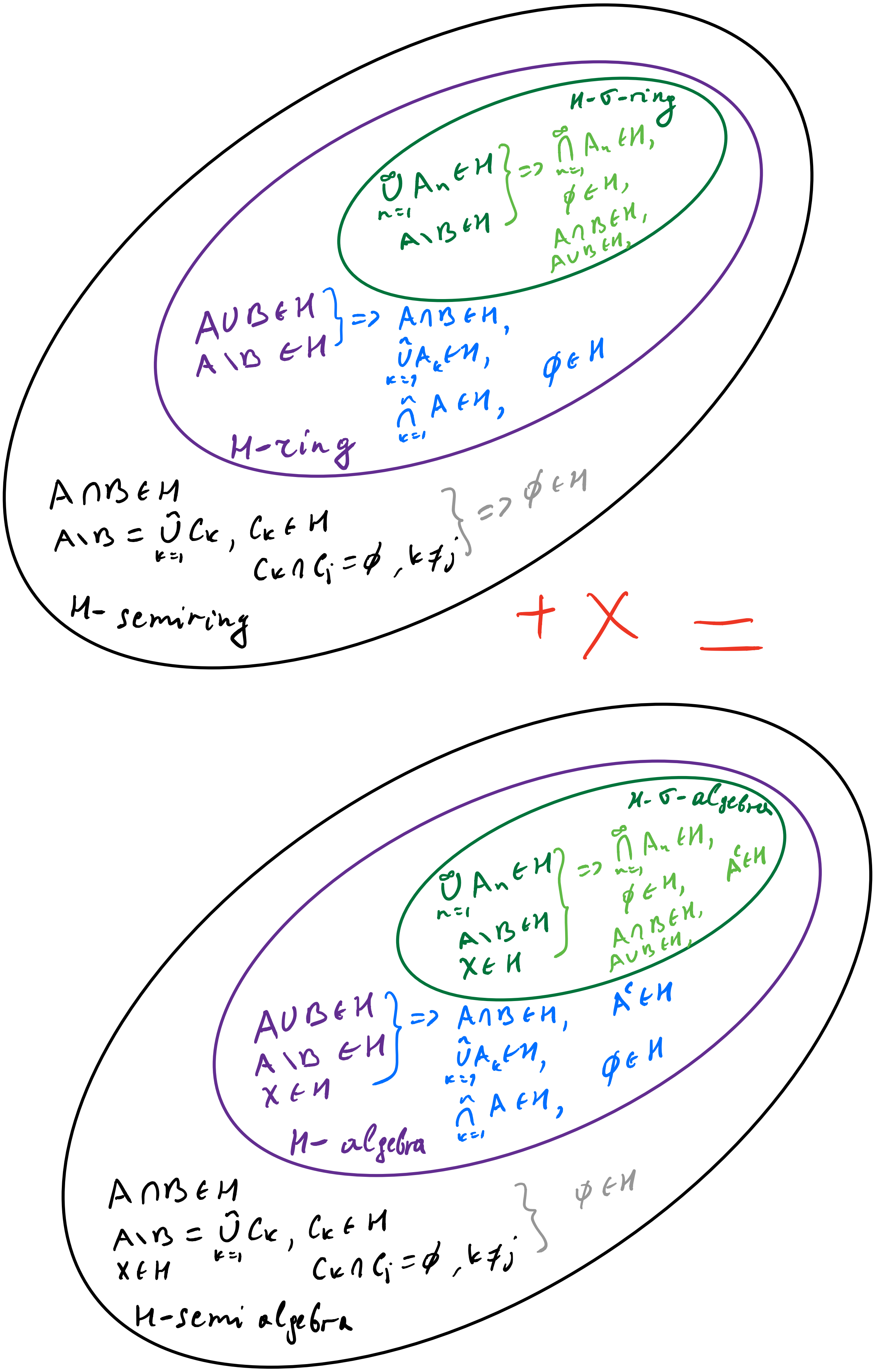
## 2. Generated classes of sets.

Let $H$ be a class of subsets of $X$.

**Def 2.7** • The smallest $\sigma$-algebra which contains the class $H$ is called the (smallest) $\sigma$-algebra generated by $H$. It is denoted by $\sigma(H)$.

• The same definition is for the ring $r(H)$, the algebra $a(H)$, and the $\sigma$-ring $\sigma r(H)$ generated by $H$.

# Relationsheeps between classes



Example 2.8 We take $X = \{a, b, c\}$

$H = \{a, b\}$

a) $\sigma(H) = \{\emptyset, X, \{a,b\}, \{c\}\}$.

We have other $\sigma$-algebras containing $H$.

e.g. $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{c,b\}\}$
$= 2^X$ but it is not smallest one.

Remark that $\sigma(H) = \alpha(H)$ here.

b) $\sigma\tau(H) = \{\emptyset, \{a,b\}\} = \tau(H)$.

Theorem 2.9 The $\sigma$-algebra generated by $H$ always exists

Proof We construct

$$\sigma(H) = \{A : A \text{ belongs to every } \sigma\text{-algebra containing } H\}$$

$$= \bigcap_{\substack{\mathcal{A}-\sigma\text{-algebra} \\ H \subset \mathcal{A}}} \mathcal{A}$$

Then $\sigma(\mathcal{H})$ is $\sigma$-algebra. Indeed, if $A_1, \dots, A_n, \dots \in \sigma(\mathcal{H})$. Then $A_1, \dots, A_n, \dots$ belongs to every $\sigma$-alg. containing $\mathcal{H}$. that is if $\mathcal{A}$ is a $\sigma$-algebra containing $\mathcal{H}$, then $A_1, \dots A_n, \dots \in \mathcal{A}$. But then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Consequently $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ ($\forall$ $\sigma$-alg. $\mathcal{A}$ containing $\mathcal{H}$). Hence, $\bigcup_{n=1}^{\infty} A_n \in \sigma(\mathcal{H})$.

Similarly, we can show that

$$A \in \sigma(\mathcal{H}) \implies A^c \in \sigma(\mathcal{H}),$$

and $X \in \sigma(\mathcal{H})$.

Prop. 2.2. implies that $\sigma(\mathcal{H})$ is a $\sigma$-algebra. It is trivial that $\sigma(\mathcal{H})$ is the smallest one. $\blacksquare$

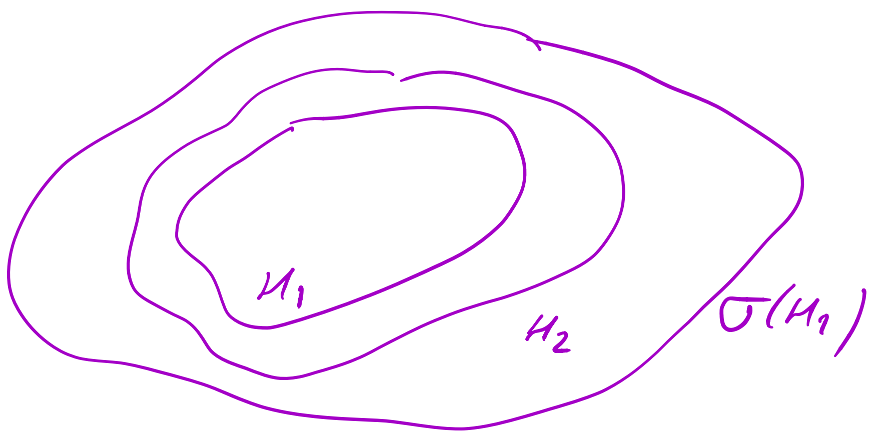
**Remark 2.10** The same statement is true for $a(\mathcal{H})$, $\tau(\mathcal{H})$, $\sigma\tau(\mathcal{H})$.

**Theorem 2.11** Let $\mathcal{H}$ be a semiring.
Then
$$r(\mathcal{H}) = \left\{ \bigcup_{k=1}^{n} A_k \ : \ A_1, \dots, A_n \in \mathcal{H}, \ n \geq 1 \right\}$$

**Corollary 2.12** Let $\mathcal{H}$ be a semialgebra.
Then
$$a(\mathcal{H}) = \left\{ \bigcup_{k=1}^{n} A_k : A_1, \dots, A_n \in \mathcal{H}, \ n \geq 1 \right\}$$

**Example 2.13** If $\mathcal{H} = \{ [a,b), -\infty < a < b < \infty \} \cup \{\emptyset\}$
Then $r(\mathcal{H}) = \left\{ A = \bigcup_{k=1}^{n} [a_k, b_k), \ \underset{k=1,\dots,n,\ n\geq 1}{-\infty < a_k < b_k < \infty} \right\}.$

**Exercise 2.14** Let $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \sigma(\mathcal{H}_1)$.
Show that $\sigma(\mathcal{H}_1) = \sigma(\mathcal{H}_2)$.




$\Rightarrow \sigma(\mathcal{H}_2) = \sigma(\mathcal{H}_1)$

**Solution** We first remark that
$$\mathcal{H}_1 \subseteq \mathcal{H}_2 \Rightarrow \mathcal{H}_1 \subseteq \sigma(\mathcal{H}_2)$$

so, $\sigma(\mathcal{H}_2)$ is a $\sigma$-algebra which contains $\mathcal{H}_1$

$\Rightarrow$ $\sigma(\mathcal{H}_1) \subseteq \sigma(\mathcal{H}_2)$ because $\sigma(\mathcal{H}_1)$ is the smallest $\sigma$-alg. which contains $\mathcal{H}_1$.

We also know that

$$\mathcal{H}_2 \subseteq \sigma(\mathcal{H}_1)$$

Similarly $\sigma(\mathcal{H}_2) \subseteq \sigma(\mathcal{H}_1)$.

Hence $\sigma(\mathcal{H}_1) = \sigma(\mathcal{H}_2)$.

## 3. Borel sets.

In this section, we will assume that $X = \mathbb{R}^d$. Let

$$\mathcal{H} = \{ [a_1, b_1) \times \dots \times [a_d, b_d) : -\infty < a_k < b_k < +\infty \} \cup \{\emptyset\}$$

We know from Lecture 1 that $\mathcal{H}$ is semiring.

**Def 2.15** $\sigma$-algebra $\mathcal{B}(\mathbb{R}^d) := \sigma(\mathcal{H})$ is called *the Borel $\sigma$-algebra*.

Sets from $\mathcal{B}(\mathbb{R}^d)$ are called *Borel sets*.

**Remark 2.16** The Borel σ-algebra contains all rectangles, all sets which can be obtained from rectangles by countable number of operations $\cap, \cup, \setminus$, taking of complement.

**Example 2.17** Let $X = \mathbb{R}$.

1) $\{a\} \in \mathcal{B}(\mathbb{R}) \quad \forall a \in \mathbb{R}$

$$\{a\} = \bigcap_{n=1}^{\infty} [a, a + \tfrac{1}{n})$$

2) $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$

$$\mathbb{Q} = \underbrace{\bigcup_{a \in \mathbb{Q}}}_{\text{countable}} \{a\}$$

3) $[a, b] \in \mathcal{B}(\mathbb{R})$

$$[a, b] = \bigcap_{n=1}^{\infty} [a, b + \tfrac{1}{n})$$

4) $(a, b) \in \mathcal{B}(\mathbb{R})$

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \tfrac{1}{n}, b)$$

5) any open set $G \subseteq \mathbb{R}^d$ belongs to $\mathcal{B}(\mathbb{R})$

$$G = \bigcup_{k=1}^{\infty} (a_k, b_k) \quad (\text{check this!})$$

6) any closed set $F \in \mathcal{B}(\mathbb{R})$ since $F^c$ is open.

**Lemma 2.18** Let $\tilde{\mathcal{H}} = \{ A \subseteq \mathbb{R}^d : A \text{ is open} \}$

Then $\sigma(\tilde{\mathcal{H}}) = \mathcal{B}(\mathbb{R}^d)$, that is,

the Borel $\sigma$-algebra is generated by all open subsets of $\mathbb{R}^d$.

**Proof.** By Example 2.17 5) (which is true for any dim. $d$)

$$\tilde{\mathcal{H}} \subseteq \mathcal{B}(\mathbb{R}^d)$$

Hence $\sigma(\tilde{\mathcal{H}}) \subseteq \mathcal{B}(\mathbb{R}^d)$.

Next, we remark that

$$[a_1, b_1) \times \dots \times [a_d, b_d) = \bigcap_{n=1}^{\infty} \left( \left( a_1 - \tfrac{1}{n}, b_1 \right) \times \dots \times (a_d, b_d) \right)$$

So, $\mathcal{H} \subseteq \sigma(\tilde{\mathcal{H}}) \Rightarrow \underbrace{\sigma(\mathcal{H})}_{= \mathcal{B}(\mathbb{R}^d)} \subseteq \sigma(\tilde{\mathcal{H}})$.

Hence, $\mathcal{B}(\mathbb{R}^d) = \sigma(\tilde{\mathcal{H}})$ ∎