1. Main classes of sets

1. Jordan measure.

Let A be a subset of $\mathbb{R}^d$. How can we define the volume of A?

If A is a rectangle

$$A = [a_1, b_1] \times \dots \times [a_d, b_d] =$$

$$= \{ x = (x_k)_{k=1}^{d} : a_k \le x_k \le b_k, \ k = 1, \dots, n \}$$

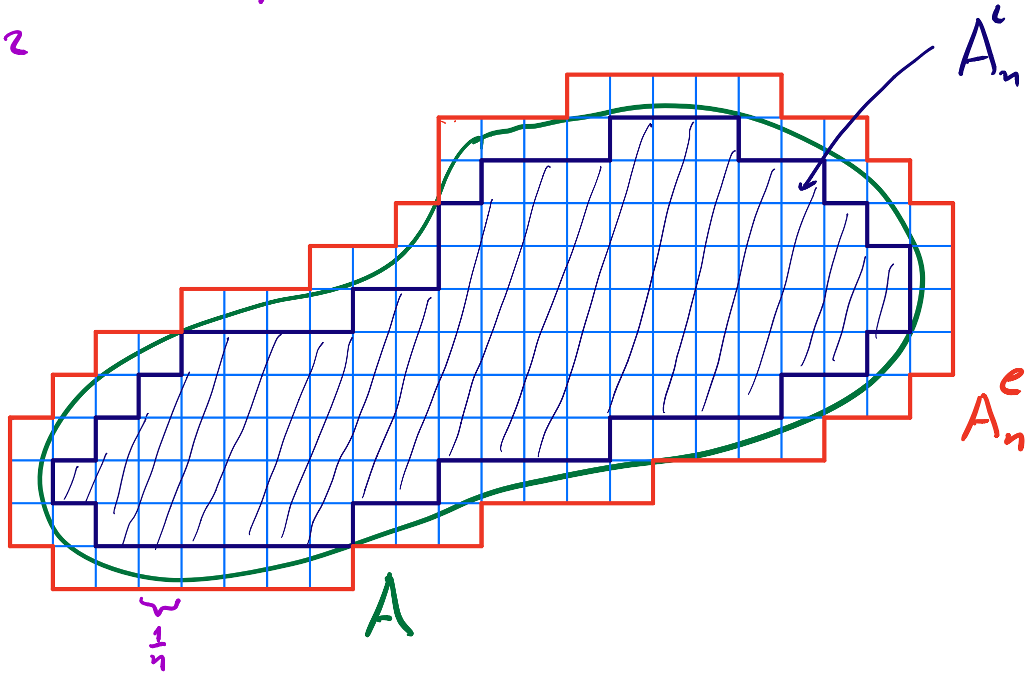
Then

$$V(A) = (b_1 - a_1) \dots (b_d - a_d) = \prod_{k=1}^{n} (b_k - a_k)$$

what if A is more general.

Then we can define the volume of A as follows:

$A \subseteq \mathbb{R}^2$



if $\lim_{n\to\infty} V(A_n^i) = \lim_{n\to\infty} V(A_n^e)$,

then we can say that the volume of A exists and equals

$$V(A) = \lim_{n\to\infty} V(A_n^i).$$

Def 1.1 $V(A)$ is called the Jordan measure of A.

Remark 1.2 The Jordan measure was defined in Lecture 2 Math 3 as

$$V(A) = \mu(A) = \int_I \mathbb{I}_A(x)\,dx = \int_A dx$$

where $I \supseteq A$ is a rectangle,

$$\mathbb{I}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

There also was shown that this two definitions are equivalent.

So, we can compute volume of more general sets. But is this definition „good enough". Does it satisfy „intuitive" properties of the volume.

Let A, B has volume (are Jordan measurable or shortly J.m.)

1) $A \cup B$ is J.m. and

$$V(A \cup B) = V(A) + V(B) \text{ if } A \cap B = \emptyset$$

2) $A \setminus B$ is J.m.

$$V(A \setminus B) = V(A) - V(B) \text{ if } B \subseteq A$$

3) $A \cap B$ is J.m.

4) Let $A_1, A_2, \dots, A_n, \dots$ are J.m.

It is not true that

$$\bigcup_{k=1}^{\infty} A_k = \{ x : \exists\, k \geq 1 \ x \in A_k \}$$

is not J.m. in general.

Example 1.3 Take $A = [0,1]^2 \cap \mathbb{Q}^2$ – set of all points from the box $[0,1]^2$ with rotoonal coefficients.

We know that A is countable. So,

$A = \{x_1, x_2, \dots \}$.

Moreover, A is not Jordan measurable. But one point sets $A_k = \{x_k\}$ are Jordan measurable and $V(A_k) = 0$.

So we obtain that $V(A_k) = 0$ but $V(A) = V(\bigcup_{k=1}^{\infty} A_k)$ does not exists (can not define it).

The intuitive properties of the volume says that

$$V(A) \leq \sum_{k=1}^{\infty} V(A_k) = 0.$$

$\Rightarrow V(A) = 0.$

This demonstrates that the Jordan measure is not well-defined for some sets which intuitively should have a volume.

Our goal is to define a volume (in general, measure) for wider class of sets which would satisfy the „intuitive" (expected) properties.

In particular, we expect that if we can define volume for sets $A_1, A_2, \ldots \subseteq \mathbb{R}^d$, then the volume exists for any set obtained from $A_1, A_2, \ldots$ by a countable number of operations: $\cup, \cap, \setminus$.

In next sections we will do this in general case, using the word „measure" instead of „volume".

2. Definition of main classes of sets

In this section, we are going to describe classes of sets for which we can define a measure.

Let X be a fixed set, $X \neq \emptyset$.

Notation We denote by 2^X the family of all subsets of X.

Def 1.4 • A non-empty class of sets $H \subset 2^X$ is called a semi-ring if

1) $A, B \in H \Rightarrow A \cap B \in H$

2) $A, B \in H \Rightarrow \exists n \in \mathbb{N} \quad \exists C_1, \dots, C_n \in H,$

$C_j \cap C_k = \emptyset, \ j \neq k$

$$A \setminus B = \bigcup_{k=1}^{n} C_k$$

• A class H is called a semi-algebra if H is a semi-ring and $X \in H$.

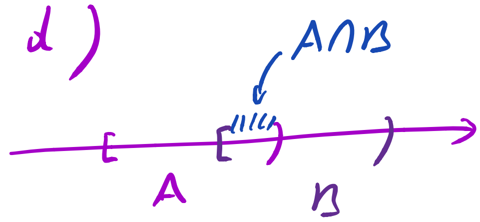
Remark 1.5 A semiring usually contains „simple" sets where a measure can be easily defined.

Example 1.6 Let $X = \mathbb{R}$,

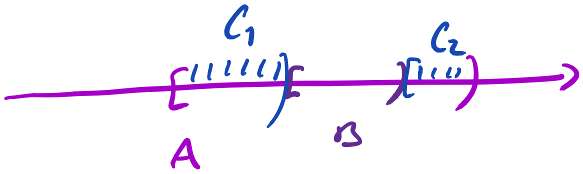
a) $H_1 = \{ [a, b), \ -\infty < a < b < +\infty \} \cup \{\emptyset\}$ is a semiring.

$A = [a, b), \quad B = [c, d)$

1) Then $A \cap B \in H_1$



2) $A \setminus B = \bigcup_{k=1}^{n} C_k$. Indeed



$A \setminus B = C_1 \cup C_2$.

b) $H_2 = \{ [a, b), \ -\infty < a < b < +\infty \} \cup \{\emptyset, \mathbb{R}\}$

$\cup \{(-\infty, b),\ b < \infty\} \cup \{[a, +\infty),\ -\infty < a\}$

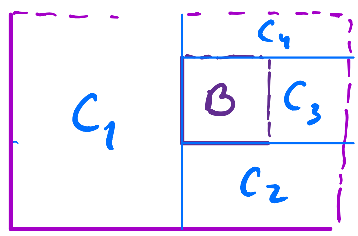
is a semialgebra •

Example 1.7 $X = \mathbb{R}^2$

a) $H_1 = \left\{ [a_1, b_1) \times [a_2, b_2) : \begin{matrix} -\infty < a_1 < b_1 < +\infty \\ -\infty < a_2 < b_2 < +\infty \end{matrix} \right\} \cup \{\emptyset\}$

is a semiring. In this case

$A \setminus B$:



$A \setminus B = C_1 \cup C_2 \cup C_3 \cup C_4$

b) $H_2 = \{ \dots \}$ can be defined by the same way as in Example 1.6, that is, the ends could be $-\infty$ and $-\infty$.

H_2 is a semialgebra •

One can see that the volume can be easily defined on sets from H_1 from examples 1.6 and 1.7.

Def. 1.8 • A non-empty class $\mathcal{H} \subset 2^X$ is called a **ring** if

(i) $A, B \in \mathcal{H} \Rightarrow A \cup B \in \mathcal{H}$

(ii) $A, B \in \mathcal{H} \Rightarrow A \setminus B \in \mathcal{H}$

• A class $\mathcal{H}$ is said to be an **algebra** if $\mathcal{H}$ is a ring and $X \in \mathcal{H}$.

Exersice 1.9 Let $\mathcal{H}$ be a ring (algebra). Show that $\mathcal{H}$ is a semiring (semialgebra, respectively).

Exersice 1.10 Let $\mathcal{H}$ be a ring. Show that

a) $\emptyset \in \mathcal{H}$;

b) $A, B \in \mathcal{H} \Rightarrow A \cap B \in \mathcal{H}$;

c) $A_1, \dots, A_n \in \mathcal{H} \Rightarrow \bigcup_{k=1}^{n} A_k \in \mathcal{H}, \ \bigcap_{k=1}^{n} A_k \in \mathcal{H}$.

Proposition 1.11 A non-empty class $\mathcal{H}$ is an algebra if and only if

1) $A, B \in \mathcal{H} \Rightarrow A \cup B \in \mathcal{H}$

2) $A \in \mathcal{H} \Rightarrow A^c = X \setminus A \in \mathcal{H}$

Proof $\Rightarrow$) Follows directly from the definition from an algebra.

Indeed, we need to check only 2)

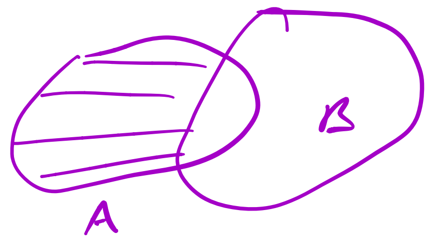
$X \in \mathcal{H}$, $A \in \mathcal{H}$. Then by (ii) of Def 1.8

$$A^c = X \setminus A \in \mathcal{H}.$$

$\Leftarrow$) we need to check only (ii) of Def 1.8.

Take $A, B \in \mathcal{H}$.

$$A \setminus B = A \cap B^c =$$



$$= (A \cap B^c)^{cc} = (\underbrace{\underbrace{A^c}_{\in \mathcal{H}} \cup \underbrace{B}_{\in \mathcal{H}}}_{\in \mathcal{H}})^c.$$

Remark that $X = \underbrace{A}_{\in \mathcal{H}} \cup \underbrace{A^c}_{\in \mathcal{H}} \in \mathcal{H}$.

$\blacksquare$