Elements of General Measure Theory and Integral

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Chapter 1

Main classes of sets

1.1 Semiring and Semialgebra

Let X be a fixed nonempty set. Assume that $X \neq \emptyset$. We next consider different classes of subsets of the set X. The set X is called a **fundamental** set.

Notation 1.1.1. 2^X denotes the family of all subsets of the set X including X and $\emptyset$.

Definition 1.1.2. A nonempty class of sets $H \subset 2^X$ is called a **semiring** if

(i) $\{A,B\} \subset H \Longrightarrow A \cap B \in H$;

(ii) $\{A,B\} \subset H \Longrightarrow \exists n \in \mathbb{N}\ \exists \{C_1,\ldots,C_n\} \subset H,\ C_j \cap C_k = \emptyset,\ j \neq k:$

$$A \setminus B = \bigcup_{k=1}^{n} C_k.$$

A class H is called a **semialgebra** if H is a semiring and $X \in H$.

Exercise 1.1.3. Let H be a semiring. Prove that $\emptyset \in H$.

Exercise 1.1.4. Prove that the following class H is a semialgebra:
a) $H = 2^X$; b) $H = \{\emptyset, X\}$; c) $H = \{\emptyset, A, A^c, X\}$, $A \subset X$.

Exercise 1.1.5. Let $X = \mathbb{R}$ and $H = \{[a,b):\ -\infty < a < b < +\infty\} \cup \{\emptyset\}$. Prove that H is a semiring.

Exercise 1.1.6. Let $a_1 < b_1$, $a_2 < b_2$ be fixed numbers, $X = [a_1,b_1) \times [a_2,b_2)$ and

$$H = \{[\alpha_1,\beta_1) \times [\alpha_2,\beta_2):\ a_k \leq \alpha_k < \beta_k < b_k,\ k = 1,2\} \cup \{\emptyset\}.$$

Show that H is a semiring.

Exercise 1.1.7. Let $H_k \subset 2^{X_k}$ for $k = 1,2$, be a semiring. Prove that the class of sets

$$H_1 \times H_2 := \{A_1 \times A_2:\ A_k \in H_k,\ k = 1,2\}$$

is a semiring of subsets in the Cartesian product of $X_1 \times X_2$.

Exercise 1.1.8. Prove that the union of two sets from a semiring does not necessarily belong to this semiring.

1.2 Ring and algebra

Definition 1.2.1. A nonempty class of sets $H \subset 2^X$ is called a **ring** if

(i) $\{A,B\} \subset H \Longrightarrow A \cup B \in H$; (ii) $\{A,B\} \subset H \Longrightarrow A \setminus B \in H$.

A class H is said to be an **algebra** if H is a ring and $X \in H$.

Exercise 1.2.2. Let H be a ring. Prove that:
a) $\emptyset \in H$; b) $\{A,B\} \subset H \Longrightarrow A \cap B \in H$;
c) $\{A_1,\dots,A_n\} \subset H \Longrightarrow \bigcup_{k=1}^n A_k \in H$ and $\bigcap_{k=1}^n A_k \in H$.

Exercise 1.2.3. Check that a ring of subsets is a semiring.

Exercise 1.2.4. Prove the following statements.
a) The class of all Jordan measurable subsets of $X = \mathbb{R}^2$ is a ring.
b) The class of all Jordan measurable subsets of $X = [0,1]^2$ is an algebra.

Exercise 1.2.5. Prove that a nonempty class of sets $H \subset 2^X$ is a ring if and only if H is a semiring and $\{A,B\} \subset H \Longrightarrow A \cup B \in H$.

Exercise 1.2.6. Let H be an algebra and $A \in H$. Show that $A^c \in H$.

Exercise 1.2.7. Prove that a nonempty class of sets $H \subset 2^X$ is an algebra if and only if

$$\{A,B\} \subset H \Longrightarrow A \cup B \in H \text{ and } A \in H \Longrightarrow A^c \in H.$$

Exercise 1.2.8. Let E be a class of subsets of X such that for any distinct sets $A,B \in E$ the equality $A \cap B = \emptyset$ holds. Set

$$H := \left\{ \bigcup_{k=1}^{n} A_k : \ n \in \mathbb{N}, \ \{A_1,\dots,A_n\} \subset E \right\} \cup \{\emptyset\}.$$

Prove that H is a ring.

1.3 σ-ring and σ-algebra

Definition 1.3.1. A nonempty class of sets $H \subset 2^X$ is called a **σ-ring** if

(i) $\{A_1,A_2,\dots,A_n,\dots\} \subset H \Longrightarrow \bigcup_{n=1}^{\infty} A_n \in H$;

(ii) $\{A,B\} \subset H \Longrightarrow A \setminus B \in H$.

A class H is said to be a **σ-algebra** if H is a σ-ring and $X \in H$.

Exercise 1.3.2. Check that the classes 2^X and $\{\emptyset, X\}$ are σ-algebras.

Exercise 1.3.3. Check that a σ-ring is a ring.

Exercise 1.3.4. Let H be a σ-ring. Prove that

$$\{A_1, A_2, \ldots, A_n, \ldots\} \subset H \Longrightarrow \bigcap_{n=1}^{\infty} A_n \in H.$$

Hint: Consider the set $A_1 \setminus (\bigcup_{n=2}^{\infty}(A_1 \setminus A_n))$.

Exercise 1.3.5. A set $A \subset \mathbb{R}^2$ is called **symmetric** if $(x_1, x_2) \in A \Longrightarrow (-x_1, -x_2) \in A$. We assume that the empty set is symmetric. Prove that the class of symmetric subsets of $\mathbb{R}^2$ is a σ-algebra.

Exercise 1.3.6.* Prove that there exists no σ-algebra consisting of a countable number of elements.

Exercise 1.3.7. Let $H_k \subset 2^{X_k}$, $k = 1, 2$, be σ-rings and $H_1 \times H_2 := \{A_1 \times A_2 : \ A_k \in H_k \ \ k = 1, 2\}$. Prove that the class $H_1 \times X_2$ is a semiring of subsets from $X_1 \times X_2$. Give an example which shows that the class $H_1 \times H_2$ is not always a ring.

1.4 Monotone class

Definition 1.4.1. A sequence of sets $\{A_n, \ n \geq 1\}$ is called **monotone increasing** if $A_n \subset A_{n+1}$, $n \geq 1$. In that case, $\lim_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} A_n$.

A sequence of sets $\{A_n, \ n \geq 1\}$ is called **monotone decreasing** if $A_n \supset A_{n+1}$, $n \geq 1$. In that case, $\lim_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} A_n$.

Sequences which monotone increase or decrease are called **monotone**.

Exercise 1.4.2. Show that $\lim_{n\to\infty}[0, n] = [0, +\infty); \ \lim_{n\to\infty}[n, +\infty) = \emptyset$.

Definition 1.4.3. A nonempty class of sets $H \subset 2^X$ is said to be a **monotone class** if for every monotone sequence $\{A_n, \ n \geq 1\} \subset H$ the set $\lim_{n\to\infty} A_n$ belongs to H.

Exercise 1.4.4. Prove that a σ-ring is a monotone class.

Exercise 1.4.5. Let $X = \mathbb{R}$ and

$$H := \{[m, n] : \ \{m, n\} \subset \mathbb{Z}, \ m < n\} \cup \{(-\infty, n] : \ n \in \mathbb{Z}\} \cup \{[n, +\infty) : \ n \in \mathbb{Z}\} \cup \{\mathbb{R}\}.$$

Check that H is a monotone class.

Theorem 1.4.6. *A monotone ring is a σ-ring.*

Proof. Let H be a ring and a monotone class. Then Condition (ii) of Definition 1.3.1 is satisfied. Let $\{A_n,\ n \ge 1\} \subset H$. Since H is a ring, we have

$$\forall m \ge 1: \quad \bigcup_{k=1}^{m} A_k \in H.$$

Moreover,

$$\forall m \ge 1: \quad \bigcup_{k=1}^{m} A_k \subset \bigcup_{k=1}^{m+1} A_k.$$

Since H is a monotone class,

$$\lim_{m\to\infty} \left(\bigcup_{k=1}^{m} A_k \right) \in H \Longleftrightarrow \bigcup_{m=1}^{\infty} \left(\bigcup_{k=1}^{m} A_k \right) = \bigcup_{m=1}^{\infty} A_m \in H.$$

Consequently, Condition (i) of Definition 1.3.1 also holds. □

1.5 Minimal classes of sets

1.5.1 Minimal ring, algebra, σ-ring, σ-algebra, monotone class containing a given class of sets

Let X be a fundamental set and H be a class of subsets of X.

Definition 1.5.1. The following class of sets

$$r(H) := \bigcap_{K \text{ is ring},\ K \supset H} K$$

is called the **ring generated by the class** H or the **minimal ring containing the class** H.

Remark 1.5.2. Rings containing the class H exists. For instance, the class 2^X is a ring and $2^X \supset H$.

Lemma 1.5.3. *The intersection of any family of rings is also a ring.*

Proof. Let $\{K_t:\ t \in T\}$ be a family of rings. Then

$$\begin{aligned} \{A,B\} \subset \bigcap_{t\in T} K_t &\Longrightarrow \forall t \in T: \quad \{A,B\} \subset K_t \\ &\Longrightarrow \forall t \in T: \quad \{A\cup B,\ A\setminus B\} \subset K_t \Longrightarrow \{A\cup B,\ A\setminus B\} \subset \bigcap_{t\in T} K_t. \end{aligned}$$

Thus, the class $\bigcap_{t\in T} K_t$ is a ring. □

Lemma 1.5.3 implies the correctness of Definition 1.5.1, i.e. that the class $r(H)$ is a ring.

Exercise 1.5.4. Prove that statements similar to Lemma 1.5.3 are true for:

a) algebra; b) σ-ring; c) σ-algebra; d) monotone class.

Exercise 1.5.5. Show that the intersection of semirings is not necessarily a semiring.

Definition 1.5.6. The following classes of sets

$$a(H) := \bigcap_{G \text{ is algebra}, \ G \supset H} G, \quad \sigma r(H) := \bigcap_{G \text{ is } \sigma\text{-ring}, \ G \supset H} G,$$
$$\sigma a(H) := \bigcap_{G \text{ is } \sigma\text{-algebra}, \ G \supset H} \quad m(H) := \bigcap_{G \text{ is monotone class}, \ G \supset H} G$$

are called the **algebra** $a(H)$, the **σ-ring** $\sigma r(H)$, the **σ-algebra** $\sigma a(H)$ and the **monotone class** $m(H)$ **generated by** H, respectively.

The classes $a(H)$, $\sigma r(H)$, $\sigma a(H)$ and $m(H)$ are also called the **minimal algebra**, the **minimal σ-ring**, the **minimal σ-algebra** and the **minimal monotone class** containing H, respectively.

Exercise 1.5.7. Let X be a finite set and $H = \{\{x\} : \ x \in X\}$. Show that $r(H) = a(H) = \sigma r(H) = \sigma a(H) = 2^X$.

Exercise 1.5.8. Prove that

a) $H_1 \subset H_2 \subset a(H_1) \Longrightarrow a(H_1) = a(H_2)$;

b) $H_1 \subset H_2 \subset \sigma a(H_1) \Longrightarrow \sigma a(H_1) = \sigma a(H_2)$.

Exercise 1.5.9. Let a set $B \subset X$ be fixed. Prove that $\sigma r(H \cap B) = \sigma r(H) \cap B$. Here $E \cap B := \{A \cap B : \ A \in E\}$ for a class of sets E.

Hint: Check that $\sigma r(H) \cap B \supset H \cap B$ and $\sigma r(H) \cap B$ is a σ-ring.

Exercise 1.5.10. Show that $\sigma a(\sigma a(H)) = \sigma a(H)$.

Theorem 1.5.11. *Let H be a semiring. Then*

$$r(H) = \left\{ \bigcup_{k=1}^{n} A_k : \ n \in \mathbb{N}, \ \{A_1, \ldots, A_n\} \subset H \right\}.$$

Proof. Let $M := \{\bigcup_{k=1}^{n} A_k : \ n \in \mathbb{N}, \ \{A_1, \ldots, A_n\} \subset H\}$. Then we have $H \subset M \subset r(H)$. Let us prove that the class M is a ring. Indeed, for sets $\{A, B\} \subset M$ the set $A \cup B$ belongs to M according to the definition of the class M. If $\{A, B\} \subset M$, then

$$A = \bigcup_{k=1}^{n} A_k, \quad B = \bigcup_{j=1}^{m} B_j, \quad \{A_1, \ldots, A_n; B_1, \ldots, B_m\} \subset H,$$

and

$$A \setminus B = \left(\bigcup_{k=1}^{n} A_k \right) \setminus \left(\bigcup_{j=1}^{m} B_j \right) = \bigcup_{k=1}^{n} \bigcap_{j=1}^{m} (A_k \setminus B_j).$$

Since H is a semiring, one can assume that

$$A_k \cap A_j = \emptyset, \quad B_k \cap B_j = \emptyset, \quad k \neq j.$$

Moreover,

$$A_k \setminus B_j = \bigcup_{r=1}^{l} C_{kjr}, \quad \{C_{kjr}\} \subset H, \quad l = l(k,j); \quad C_{kjr} \cap C_{kjs} = \emptyset, \quad r \neq s.$$

Thus, $A \setminus B = \bigcup_{k=1}^{n} \bigcap_{j=1}^{m} \bigcup_{r=1}^{l} C_{kjr}$. □

Exercise 1.5.12. Let $H = \{A_1, A_2, \ldots, A_n\} \subset X$. Prove that:

a) $a(H)$ consists of at most 2^{2^n} sets; b) $a(H) = \sigma a(H)$.

Hint: Consider sets of the form $\hat{A}_1 \cap \hat{A}_2 \cap \cdots \cap \hat{A}_n$, where $\hat{A}_k$ equals A_k or $X \setminus A_k$ for every $1 \leq k \leq n$.

Exercise 1.5.13. The **minimal semiring** $p(H)$ **containing a class** H is the semiring which contains the class H and is contained in any semiring which contains the class H. Let $H = \{(-\infty, a] : a \in \mathbb{R}\}$. Show that $p(H) = \{(a,b] : -\infty \leq a < b < +\infty\} \cup \{\emptyset\}$.

1.5.2 Borel sets

Let (X,d) be a metric space, $\mathscr{G}$ be a class of all open in (X,d) subsets of X.

Definition 1.5.14. The σ-algebra $\mathscr{B}(X) = \sigma a(\mathscr{G})$ is called the **σ-algebra of Borel sets**.

Exercise 1.5.15. Let (X,d) be a separable metric space and $H = \{B(x,r) : x \in X, r > 0\}$, where $B(x,r) := \{y \in X : d(x,y) < r\}$. Prove that $\mathscr{B}(X) = \sigma a(H)$.

Hint: Check that $H \subset \mathscr{G} \subset \sigma a(H)$.

Exercise 1.5.16.* Let (X,d) be a separable metric space. Prove that there exists a countable class of sets $H \subset 2^X$ such that $\sigma a(H) = \mathscr{B}(X)$.

Exercise 1.5.17. Let (X,d) be a separable metric space and $\mathscr{F}$ be a class of all closed in (X,d) subsets of X. Prove that

$$\mathscr{B}(X) = \sigma a(\mathscr{F}) = \sigma a\left(\{\bar{B}(x,r) : x \in X, r > 0\}\right),$$

where $\bar{B}(x,r) = \{y \in X : d(x,y) \leq r\}$.

Exercise 1.5.18. Prove that:

a) any one-point set is a Borel set;

b) any countable set is a Borel set.

Exercise 1.5.19. Let $\mathscr{B} := \mathscr{B}(\mathbb{R})$ be the σ-algebra of Borel sets on $\mathbb{R}$ with the distance $d(x,y) = |x-y|$, $\{x,y\} \subset \mathbb{R}$. Prove the the following sets are Borel:

a) the set of rational numbers $\mathbb{Q}$;

b) the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$;

c) $(a,b]$, $\{a,b\} \subset \mathbb{R}$, $a < b$;

d) The set of all real numbers whose decimal representation contains infinitely many times the digits 4.

Exercise 1.5.20. Prove that

$$\begin{aligned}\mathscr{B} &= \sigma a\left(\{(-\infty,a]:\ a\in\mathbb{R}\}\right) = \sigma a\left(\{-\infty,a]:\ a\in\mathbb{Q}\}\right)\\ &= \sigma a\left(\{(a,b]:\ -\infty<a<b<+\infty\}\right).\end{aligned}$$

Exercise 1.5.21.* Let (X,d) be a separable metric space. Prove that $\mathscr{B}(X)$ has at most continuum cardinality.

1.5.3 Monotone class and σ-ring generated by a ring

Theorem 1.5.22. *Let H be a ring of subsets of X. Then $\sigma r(H) = m(H)$.*

Proof. Since $\sigma r(H)$ is a monotone class, we have the inclusion $m(H)\subset \sigma r(H)$, according to the definition of $m(H)$.

Let us prove that $m(H)$ is a ring. For every $B\in m(H)$ we consider the following class of sets

$$L(B) := \{C\subset X:\ \{B\cup C,\ B\setminus C,\ C\setminus B\}\subset m(H)\}.$$

The following two statements hold.

(i) Since H is a ring and $H\subset m(H)$, one has $\forall A\in H:\ H\subset L(A)$.

(ii) $\forall B\in m(H):\ L(B)$ is a monotone class.

Let us prove the second statement. Let $\{C_n:\ n\geq 1\}\subset L(B)$, $C_n\subset C_{n+1}$, $n\geq 1$. Then for every $n\geq 1$ we have

$$\begin{aligned}&C_n\cup B\subset C_{n+1}\cup B,\quad C_n\setminus B\subset C_{n+1}\setminus B,\quad B\setminus C_n\subset B\setminus C_{n+1};\\ &\{B\cup C_n,\ B\setminus C_n,\ C_n\setminus B\}\subset m(H).\end{aligned}$$

Since $m(H)$ is a monotone class, we have

$$\begin{aligned}m(H) &\ni \bigcup_{n=1}^{\infty}(C_n\cup B) = \left(\bigcup_{n=1}^{\infty}C_n\right)\cup B,\\ m(H) &\ni \bigcup_{n=1}^{\infty}(C_n\setminus B) = \left(\bigcup_{n=1}^{\infty}C_n\right)\setminus B,\\ m(H) &\ni \bigcup_{n=1}^{\infty}(B\setminus C_n) = B\setminus\left(\bigcup_{n=1}^{\infty}C_n\right).\end{aligned}$$

Hence $\bigcup_{n=1}^{\infty}C_n\in L(B)$. Similarly, one can check that $\bigcap_{n=1}^{\infty}C_n in L(B)$ for a decreasing sequence $\{C_n:\ n\geq 1\}$ from $L(B)$. The statement (ii) is proved.

Since $L(A)$ is a monotone class for all $A \in H$, by (ii), and $H \subset L(A)$, by (i), we obtain

$$\begin{aligned}
&\forall A \in H: \ m(H) \subset L(A) \\
&\qquad \Longrightarrow \forall A \ \forall C_1 \in m(H): \ \{A \cup C_1, \ A \setminus C_1, \ C_1 \setminus A\} \subset m(H) \\
&\qquad \Longrightarrow H \subset L(C_1) \Longrightarrow \forall C_1 \in m(H): \ m(H) \subset L(C_1) \\
&\qquad \Longrightarrow \forall \{C_1, C_2\} \subset m(H): \ \{C_1 \cup C_2, \ C_1 \setminus C_2, \ C_2 \setminus C_1\} \subset m(H).
\end{aligned}$$

Thus, $m(H)$ is a ring. By Theorem 1.4.6, $m(H)$ is a σ-ring. So, $\sigma r(H) \subset m(H)$. Consequently, $\sigma r(H) = m(H)$. □

Exercise 1.5.23. Let H be an algebra of sets. Prove that $\sigma a(H) = m(H)$.

Chapter 2

Functions of sets. Measures

2.1 The main classes of functions of sets

Let X be a fixed nonempty set and $H \subset 2^X$ be a non-empty class of sets of X. The object of investigation of the measure theory are functions of the form

$$\mu : H \to (-\infty, +\infty)$$

which satisfy special requirements. Length, area, and volume defined for some classes of sets of the line, plane, and space, respectively, are real examples of such functions. The charge of parts of the space in an electric field is another type of example. Those examples lead to a narrow, but important for mathematics, class of functions. For instance, the area is nonnegative, the area of a figure consisting of a union of two nonintersecting parts is equal to the sum of areas of those parts and so on. The special requirements for functions of sets mentioned above particularly consist in the transfer of properties of real functions of sets to an abstract situation and particularly are related to the mathematical necessity.

We will further consider functions taking the value $+\infty$. For example, it is natural to assume that the length of the real line equal $+\infty$. We will assume that

$$(+\infty) + (+\infty) = +\infty; \quad \forall a \in \mathbb{R} : \ a < +\infty, \ a + \infty := +\infty + a := +\infty.$$

Definition 2.1.1. A function $\mu : H \to (-\infty, +\infty]$ is called:

(i) **nonnegative**, if $\forall A \in H : \quad \mu(A) \geq 0$;

(ii) **finitely semiadditive**, (or simply **semiadditive**) if

$$\forall n \in \mathbb{N} \ \forall \{A_1, \ldots, A_n\} \subset H, \ \bigcup_{k=1}^{n} A_k \in H : \quad \mu\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} \mu(A_k);$$

(iii) **finitely additive** (or simply **additive**) if

$$\forall n \in \mathbb{N} \quad \forall \{A_1, \ldots, A_n\} \subset H, \ \bigcup_{k=1}^{n} A_k \in H, \ A_j \cap A_k = \emptyset, \ j \neq k :$$
$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k);$$

(iv) **countably semiadditive** (or **σ-semiadditive**), if

$$\forall \{A_n : \ n \geq 1\} \subset H, \ \bigcup_{n=1}^{\infty} A_n \in H : \quad \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n);$$

(v) **countably additive** (or **σ-additive**), if

$$\forall \{A_n : \ n \geq 1\} \subset H, \ \bigcup_{n=1}^{\infty} A_n \in H, \ A_j \cap A_k = \emptyset, \ \ j \neq k :$$
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n);$$

(vi) **monotone**, if $\forall \{A, B\} \subset H, \ A \subset B : \quad \mu(A) \leq \mu(B);$

(vii) **finite**, if $\forall A \in H : \quad \mu(A) < +\infty;$

(viii) **σ-finite**, if

$$\exists \{A_n : \ n \geq 1\} \subset H : \quad \bigcup_{n=1}^{\infty} A_n = X \ \text{ and } \ \forall n \geq 1 : \ \mu(A_n) < +\infty.$$

Exercise 2.1.2. Assume that $\emptyset \in H$, a function μ is additive and exists a set $A \in H$ such that $\mu(A) < +\infty$. Prove that $\mu(\emptyset) = 0$.

Exercise 2.1.3. Assume that $\emptyset \in H$, $\mu(\emptyset) = 0$ and μ is σ-additive on H. Prove that μ is additive on H.

Hint: Use the equality $A \cup B = A \cup B \cup \emptyset \cup \cdots \cup \emptyset \cup \ldots$.

Remark 2.1.4. We will not consider functions μ which take the value $+\infty$ at **every** set from H.

2.2 Measures. Basic properties of measures

Definition 2.2.1. A nonnegative σ-additive function defined on a semiring is called a **measure**.

Exercise 2.2.2. Let μ be a measure. Prove that $\mu(\emptyset) = 0$.

Exercise 2.2.3. Prove that a measure is an additive function.

Exercise 2.2.4.* Let $X = \{x_1, x_2, \dots, x_n, \dots\}$ and $H = 2^X$. For a family of nonnegative numbers p_n, $n \geq 1$, satisfying $\sum_{n=1}^{\infty} p_n = 1$ define $\mu(A) := \sum_{n:\, x_n \in A} p_n$, $A \in H$. Prove that μ is a measure on H.

Exercise 2.2.5. Let $X = [0,1]^2$, H be an algebra of all Jordan measurable subsets of X and the function μ is the Jordan measure on H. Check that μ is a nonnegative additive function on H.

Theorem 2.2.6. *Let R be a ring and μ be a measure on R. Then*

1) μ is monotone on R;

2) $\forall \{A,B\} \subset R,\ A \subset B,\ \mu(A) < +\infty$:

$$\mu(B \setminus A) = \mu(B) - \mu(A);$$

3) If $\{A,B\} \subset R$ and at least one of the values $\mu(A)$, $\mu(B)$ is finite, then

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B);$$

4) If $\{A, B_1, \dots, B_n\} \subset R$ and $A \subset \bigcup_{k=1}^{n} B_k$, then

$$\mu(A) \leq \sum_{k=1}^{n} \mu(B_k);$$

5) μ is σ-semiadditive on R.

Proof. 1) Let $\{A,B\} \subset R$ and $A \subset B$. Then

$$B = A \cup (B \setminus A), \quad A \cap (B \setminus A) = \emptyset.$$

Using the additivity and the nonnegativity of the measure μ, one has

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A). \tag{2.2.1}$$

2) If $\mu(A) < +\infty$, then equality (2.2.1) yields

$$\mu(B \setminus A) = \mu(B) - \mu(A).$$

3) If $\mu(A) < +\infty$ and $\mu(B) < +\infty$, then $\mu(A \cap B) < +\infty$, according to 1). Moreover,

$$A \cup B = (A \setminus (A \cap B)) \cup B, \quad (A \setminus (A \cap B)) \cap B = \emptyset.$$

Hence, using the additivity of the measure μ and 2), we have

$$\mu(A \cup B) = \mu(A \setminus (A \cap B)) + \mu(B) = \mu(A) - \mu(A \cap B) + \mu(B).$$

4) By 1) and the additivity of μ, we have

$$\mu(A) \le \mu\left(\bigcup_{k=1}^{n} B_k\right) = \mu\left(B_1 \cup (B_2 \setminus B_1) \cup (B_3 \setminus (B_1 \cup B_2)) \cup \cdots \cup \left(B_n \setminus \bigcup_{k=1}^{n-1} B_k\right)\right)$$
$$= \mu(B_1) + \mu(B_2 \setminus B_1) + \mu(B_3 \setminus (B_1 \cup B_2)) + \cdots + \mu\left(B_n \setminus \left(\bigcup_{k=1}^{n-1} B_k\right)\right) \le \sum_{k=1}^{n} \mu(B_k).$$

5) Similarly to the proof of 4), we obtain

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty}\left(A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right)\right)\right) = \sum_{n=1}^{\infty} \mu\left(A_n \setminus \bigcup_{k=1}^{n-1} A_k\right) \le \sum_{n=1}^{\infty} \mu(A_n),$$

by the σ-additivity of the measure μ. Here we assume that $\bigcup_{k=1}^{0} A_k := \emptyset$. □

Remark 2.2.7. Properties 1)-4) of Theorem 2.2.6 is valid for any nonnegative and additive function μ.

Exercise 2.2.8. Prove that a nonnegative, additive and σ-semiadditive function μ on a ring R is a measure on R.

Hint: Let $\{A_n : n \ge 1\} \subset R$, $\bigcup_{n=1}^{\infty} A_n \in R$, $A_n \cap A_m = \emptyset$, $m \ne n$. From the monotonicity and additivity of μ we have

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \ge \mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k), \quad n \ge 1.$$

Exercise 2.2.9. Let μ be a measure on a σ-ring H and for $\{A_n : n \ge 1\} \subset H$ $mu(A_n) = 0$, $n \ge 1$. Prove that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0.$$

Hint: Use the σ-semiadditivity of a measure.

Exercise 2.2.10. Let μ be a measure on a σ-algebra H. Let $\mu(X) = 1$ and a family of sets $\{A_n : n \ge 1\} \subset H$ satisfy $\mu(A_n = 1)$, $n \ge 1$. Prove that

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 1.$$

Hint: Use De Morgan's law and Exercise 2.2.9.

Exercise 2.2.11. Let μ be an additive finite function of a ring R. Prove that for every sets A_1, A_2, A_3 from R the following inequality

$$\mu(A_1 \cup A_2 \cup A_3) = \mu(A_1) + \mu(A_2) + \mu(A_3)$$
$$- \mu(A_1 \cap A_2) - \mu(A_1 \cap A_3) - \mu(A_2 \cap A_3) + \mu(A_1 \cap A_2 \cap A_3)$$

holds.

Exercise 2.2.12. Let μ be a measure on an algebra $H \subset 2^X$ and $\mu(X) = 1$. Prove the following statement. If a family of sets $\{A_1, \ldots, A_n\} \subset H$ satisfies the inequality

$$\mu(A_1) + \cdots + \mu(A_n) > n - 1,$$

then

$$\mu\left(\bigcap_{k=1}^{n} A_k\right) > 0.$$

Exercise 2.2.13.* Let μ be a measure on a σ-algebra $H \subset 2^X$. Let also $\mu(X) = 1$ and a family of sets $\{A_n : \ n \geq 1\} \subset H$ satisfy

$$\sum_{n=1}^{\infty} \mu(A_n) < +\infty.$$

Consider the set

$$B := \left\{ x \in X : \begin{array}{l} x \text{ belong to a finite number of} \\ \text{sets } A_n, \ n \geq 1, \text{ or } x \notin \bigcup_{n=1}^{\infty} A_n \end{array} \right\}.$$

Prove that $B \in H$ and $\mu(B) = 1$.

Hint: Notice that $B^c = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$ and use the monotonisity and the σ-semiadditivity of the measure μ. The set B^c is the set of all points x which belongs to the infinite number of sets from $\{A_n : \ n \geq 1\}$.

2.3 Continuity of measure

Theorem 2.3.1 (Continuity from below)**.** *Let R be a ring and μ be a measure on R. Then for every increasing sequence $\{A_n : \ n \geq 1\}$ such that $\bigcup_{n=1}^{\infty} A_n \in R$ one has*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. I. If $\exists n_0 : \ \ \mu(A_{n_0}) = +\infty$, then for every $n \geq n_0$ such that $\mu(A_n) = +\infty$ we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = +\infty,$$

by the monotonicity of μ on R. Consequently, the statement holds.

II. Let $\mu(A_n) < +\infty$ for all $n \geq 1$. By the σ-additivity of μ and Property 2) of Theorem 2.2.6, we obtain

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu(A_1 \cup (A_2 \setminus A_1) \cup \cdots \cup (A_n \setminus A_{n-1}) \cup \ldots) \\
&= \mu(A_1) + \sum_{k=2}^{\infty} \mu(A_k \setminus A_{k-1}) = \mu(A_1) + \lim_{n\to\infty} \sum_{k=2}^{n} (\mu(A_k) - \mu(A_{k-1})) \\
&= \lim_{n\to\infty} \mu(A_n).
\end{aligned}$$

□

Exercise 2.3.2. Prove that a nonnegative, additive and continuous from below function on a ring is a measure.

Theorem 2.3.3. *Let R be a ring and μ is a measure on R. Then for every decreasing sequence $\{A_n :\ n \geq 1\}$ such that $\mu(A_1) < +\infty$ and $\bigcap_{n=1}^{\infty} A_n \in R$ one has*

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. According to Theorem 2.3.1, we obtain

$$\mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=2}^{\infty} (A_1 \setminus A_n)\right) = \lim_{n\to\infty} \mu(A_1 \setminus A_n).$$

Since $\mu(A_1) < +\infty$, we get

$$\mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \left(\mu(A_1) - \mu(A_n)\right),$$

by Property 2) of Theorem 2.2.6. □

Exercise 2.3.4. Let $X = \mathbb{N}$, $R = 2^{\mathbb{N}}$ and μ be a measure on R defined by the equalities $\mu(\emptyset) = 0$ and $\mu(\{k\}) = 1$, $k \in \mathbb{N}$. We consider the following sets

$$A_n = \{n, n+1, \dots\},\ \ A_n \supset A_{n+1},\ \ n \geq 1; \quad \bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Check that

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) \neq \lim_{n\to\infty} \mu(A_n).$$ [1]

Exercise 2.3.5. Prove that nonnegative and additive function defined on a ring which takes finite values and is continuous from above at the set $\emptyset$ is a measure.

Exercise 2.3.6. Give an example of a ring R and a measure μ such that there exists decreasing sequence $\{A_n :\ n \geq\} \subset R$ with $\mu(A_n) = +\infty$ satisfying the following property:

a) $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = +\infty$; b) $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$; c) $0 < \mu\left(\bigcap_{n=1}^{\infty} A_n\right) < +\infty$.

Exercise 2.3.7.* Let μ be a measure on a ring R and a sequence of sets $\{A_n :\ n \geq\} \subset R$ satisfy the following conditions

$$\mu(A_1) < +\infty, \quad \bigcap_{n=1}^{\infty} A_n \in R, \quad \forall n_1, n_2 \in \mathbb{N}\ \ \exists n_3 \in \mathbb{N} :\ \ A_{n_3} \subset A_{n_1} \cap A_{n_2}.$$

Prove that

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \inf_{n\geq 1} \mu(A_n).$$

[1] This shows that the condition $\mu(A_1) < +\infty$ is essential in Theorem 2.3.3.

Exercise 2.3.8. For any sequence $\{A_n:\ n \geq 1\}$ of subsets of a set X

$$\varliminf_{n\to\infty} A_n := \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k, \quad \varlimsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$$

are called **lower** and **upper limits** of the set $\{A_n:\ n \geq 1\}$, respectively. If

$$\varliminf_{n\to\infty} A_n = \varlimsup_{n\to\infty} A_n =: \lim_{n\to\infty} A_n,$$

then the sequence $\{A_n:\ n \geq 1\}$ is called **convergent**. Let μ be a measure on a σ-algebra $\mathscr{F}$ of subsets from X and $\{A_n:\ n \geq 1\}$ be a sequence of subsets from $\mathscr{F}$. Prove that

$$\mu\left(\varliminf_{n\to\infty} A_n\right) \leq \varliminf_{n\to\infty} \mu(A_n).$$

Under the additional condition $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) < +\infty$, prove that

$$\mu\left(\varlimsup_{n\to\infty} A_n\right) \geq \varlimsup_{n\to\infty} \mu(A_n).$$

This implies that for a convergent sequence $\{A_n:\ n \geq 1\}$ satisfying $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) < +\infty$ one has

$$\mu\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

2.4 Examples of measures

The example of a measure defined on a σ-algebra of all subsets of a countable set X from Exercise 2.2.4 is important for different fields of mathematics such as a probability theory.

In this section, we will consider other important examples of measures.

Theorem 2.4.1. *Let R be a ring of all Jordan measurable subsets of $\mathbb{R}^d$ and μ be a Jordan measure on R. Then the function μ is σ-additive on R.*

Proof. Let

$$\{A_n:\ n \geq 1\} \subset R, \quad A := \bigcup_{n=1}^{\infty} A_n \in R, \quad A_n \cap A_m = \emptyset,\ \ n \neq m.$$

I. Let $\bigcup_{n=1}^{\infty} A_n \subset A$, then

$$\mu\left(\bigcup_{n=1}^{N} A_n\right) \leq \mu(A),$$

by the monotonicity of μ on R. Since μ is additive on R,

$$\sum_{n=1}^{N} \mu(A_n) \leq \mu(A).$$

Hence

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu(A). \tag{2.4.1}$$

II. Let $\varepsilon > 0$ be a fixed number. We consider $\mathbb{R}^d$ as a metric space with the Euclidean distance. According to the construction of the Jordan measure, for a set $A \in R$ there exist a closed set $F \in R$ and an open set $G \in R$ such that

$$F \subset A \subset G, \quad \text{and} \quad \mu(G) - \mu(F) < \varepsilon.$$

Moreover,

$$\mu(A) < \mu(F) + \varepsilon. \tag{2.4.2}$$

Similarly, for every $n \geq 1$ and $A_n \in R$ there exist an empty set $G_n \in R$ such that

$$A_n \subset G_n, \quad \text{and} \quad \mu(G_n) - \mu(A_n) < \frac{\varepsilon}{2^n}. \tag{2.4.3}$$

Note that

$$F \subset A = \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} G_n.$$

This implies that the closed and bounded set F, which is a compact set, is covered by $\{G_n : n \geq 1\}$, i.e. $F \subset \bigcup_{n=1}^{\infty} G_n$. Since F is compact, there exists a number $N \in \mathbb{N}$ such that $F \subset \bigcup_{n=1}^{N} G_n$. So, this inclusion, the monotonicity and the semiadditivity of μ on R yield

$$\mu(F) \leq \mu\left(\bigcup_{n=1}^{n} A_n\right) \leq \sum_{n=1}^{N} \mu(G_n).$$

Consequently, by (2.4.3),

$$\mu(F) \leq \sum_{n=1}^{N} \left(\mu(A_n) + \frac{\varepsilon}{2^n}\right) < \sum_{n=1}^{\infty} \mu(A_n) + \varepsilon.$$

From this inequality and (2.4.2) implies that

$$\mu(A) < \sum_{n=1}^{\infty} \mu(A_n) + 2\varepsilon.$$

Since ε is any positive number, we can send ε to 0. Thus,

$$\mu(A) < \sum_{n=1}^{\infty} \mu(A_n). \tag{2.4.4}$$

Using (2.4.1), and (2.4.4), we obtain

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n).$$

□

Corollary 2.4.2. *Let $X = \mathbb{R}$. Define the sigma ring $\mathscr{P}_1$ as*

$$\mathscr{P}_1 = \{(a,b] : \ -\infty < a < b < +\infty\} \cup \{\emptyset\}.$$

Let the function μ on $\mathscr{P}_1$ be defined by the following equality

$$\mu(\emptyset) := 0, \quad \mu((a,b]) := b - a, \ \ (a,b] \in \mathscr{P}_1.$$

Then μ is a measure on $\mathscr{P}_1$.

Proof. μ is the restriction of the one-dimensional Jordan measure on $\mathscr{P}_1$. □

Corollary 2.4.3. *Let $X = \mathbb{R}^d$. Define the sigma ring $\mathscr{P}_2$ as*

$$\mathscr{P}_2 = \{(a_1,b_1] \times (a_2,b_2] : \ -\infty < a_k < b_k < +\infty, \ \ k = 1,2\} \cup \{\emptyset\}.$$

Let the function μ on $\mathscr{P}_2$ be defined by the following equality

$$\mu(\emptyset) := 0, \quad \mu((a_1,b_1] \times (a_2,b_2]) := (b_1 - a_1)(b_2 - a_2), \ \ (a_1,b_1] \times (a_2,b_2] \in \mathscr{P}_2.$$

Then μ is a measure on $\mathscr{P}_2$.

Proof. μ is the restriction of the two-dimensional Jordan measure on $\mathscr{P}_2$. □

Theorem 2.4.4. *For $X = \mathbb{R}$ and the semiring $\mathscr{P}_1$ define*

$$\lambda_F(\emptyset) := 0, \quad \lambda_F((a,b]) := F(b) - F(a), \ \ (a,b] \in \mathscr{P}_1,$$

where F is a nondecreasing and right continuous function on $\mathbb{R}$. Then the function λ_F is a measure on $\mathscr{P}_1$.

Proof. The function λ_F is nonnegative and additive on $\mathscr{P}_1$. We prove that λ_F is σ-additive on $\mathscr{P}_1$. Let

$$\{(a_n,b_n] : \ n \geq 1\} \subset \mathscr{P}_1, \quad (a_n,b_n] \cap (a_m,b_m] = \emptyset, \ \ n \neq m, \quad \bigcup_{n=1}^{\infty} (a_n,b_n] = (a,b] \in \mathscr{P}_1.$$

I. Using the definition of a semiring, we obtain

$$\forall N \geq 1 : \ \ (a,b] \setminus \bigcup_{n=1}^{N} (a_n,b_n] = \bigcup_{k=1}^{m} C_k, \quad \{C_k : \ k = 1,\ldots,m\} \subset \mathscr{P}_1, \ \ C_k \cap C_j = \emptyset, \ \ k \neq j.$$

Consequently, for each N we have

$$(a,b] = \bigcup_{n=1}^{N} (a_n,b_n] \cup \bigcup_{k=1}^{m} C_k.$$

Hence, by the additivity of λ_F on $\mathscr{P}_1$, we obtain the equality

$$\lambda_F((a,b]) = \sum_{n=1}^{N} \lambda_F((a_n,b_n]) + \sum_{k=1}^{m} \lambda_F(C_k).$$

Thus,

$$\forall N \geq 1: \quad \lambda_F((a,b]) \geq \sum_{n=1}^{N} \lambda_F((a_n,b_n]),$$

and, consequently,

$$\lambda_F((a,b]) \geq \sum_{n=1}^{\infty} \lambda_F((a_n,b_n]). \tag{2.4.5}$$

II. Since F is right continuous, we obtain

$$\begin{aligned} &\forall \varepsilon > 0 \;\; \exists a' \in (a,b): \;\; F(a') - F(a) < \varepsilon \\ &\implies \lambda_F((a,b]) - \lambda_F((a',b]) = F(b) - F(a) - (F(b) - F(a')) \\ &= F(a') - F(a) < \varepsilon; \end{aligned} \tag{2.4.6}$$

$$\begin{aligned} &\forall n \geq 1 \;\; \exists b'_n > b_n: \;\; F(b'_n) - F(b_n) < \frac{\varepsilon}{2^n} \\ &\implies \lambda_F((a_n,b'_n]) - \lambda_F((a_n,b_n]) = F(b'_n) - F(a_n) - (F(b_n) - F(a_n)) \\ &= F(b'_n) - F(b_n) < \frac{\varepsilon}{2^n}. \end{aligned} \tag{2.4.7}$$

We note that the following inclusions

$$[a',b] \subset (a,b] = \bigcup_{n=1}^{\infty} (a_n,b_n] \subset \bigcup_{n=1}^{\infty} (a_n,b'_n)$$

hold. Since $[a',b]$ is a compact set in $\mathbb{R}$,

$$\exists N \in \mathbb{N}: \quad [a',b] \subset \bigcup_{n=1}^{N} (a_n,b'_n) \subset \bigcup_{n=1}^{N} (a_n,b'_n].$$

Next the semiadditivity λ_F yields

$$\lambda_F((a',b]) \leq \sum_{n=1}^{N} \lambda_F((a_n,b'_n]) \leq \sum_{n=1}^{\infty} \lambda_F((a_n,b'_n]).$$

Using inequalities (2.4.6) and (2.4.7), we have the following inequality

$$\lambda_F((a,b]) < \lambda_F((a',b]) + \varepsilon \leq \sum_{n=1}^{\infty} \left(\lambda_F((a_n,b_n]) + \frac{\varepsilon}{2^n} \right) + \varepsilon = \sum_{n=1}^{\infty} \lambda_F((a_n,b_n]) + 2\varepsilon.$$

Making $\varepsilon \to 0+$, we obtain

$$\lambda_F((a,b]) \leq \sum_{n=1}^{\infty} \lambda_F((a_n,b_n]).$$

This together with (2.4.5) implies

$$\lambda_F((a,b]) = \sum_{n=1}^{\infty} \lambda_F((a_n,b_n]).$$

□

Exercise 2.4.5. Let $G \in \mathrm{C}(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})$, and for $X = \mathbb{R}$ and the semiring $\mathscr{P}_1$

$$\nu_G(\emptyset) := 0, \quad \nu_G((a,b]) := G(b) - G(a), \;\; (a,b] \in \mathscr{P}_1.$$

Prove that ν_G is a σ-additive function on $\mathscr{P}_1$.

Chapter 3

Extension of measures

3.1 Extension of a measure from semiring to the generated ring

Let X be a fundamental set.

Definition 3.1.1. Let $\mathscr{E}_k \subset 2^X$, $\mu_k : \mathscr{E}_k \to (-\infty, +\infty]$, $k = 1, 2$. The function μ_2 is called an **extension of the function** μ_1 (μ_1 is called the **restriction of** μ_2), if

$$\mathscr{E}_1 \subset \mathscr{E}_2, \quad \text{and} \quad \forall A \in \mathscr{E}_1 : \ \mu_1(A) = \mu_2(A).$$

Theorem 3.1.2. *Let μ be a measure on a semiring $\mathscr{P}$. The measure μ can be extended to a measure on $r(\mathscr{P})$ by a unique way. Moreover, this extension is finite (σ-finite) if μ is finite (σ-finite, resp.).*

Proof. I. Definition of the extension. For $A \in r(\mathscr{P})$ we have

$$A = \bigcup_{k=1}^{n} C_k, \quad \{C_1, \ldots, C_n\} \subset \mathscr{P}, \ C_k \cap C_j = \emptyset, \ k \neq j.$$

Set

$$\bar{\mu}(A) := \sum_{k=1}^{n} \mu(C_k).$$

The function $\bar{\mu}$ is well-defined. Indeed, let us consider other representation of A

$$A = \bigcup_{j=1}^{m} D_j, \quad \{D_1, \ldots, D_m\} \subset \mathscr{P}, \ D_k \cap D_j = \emptyset, \ k \neq j.$$

Then for any $1 \leq k \leq n$, $1 \leq j \leq m$ we have

$$C_k = C_k \cap A = \bigcup_{j=1}^{m} (C_k \cap D_j), \quad D_j = A \cap D_j = \bigcup_{k=1}^{n} (C_k \cap D_j).$$

Furthermore, the sets $\left\{C_k \cap D_j : \ 1 \le k \le n, \ 1 \le j \le m\right\} \subset \mathscr{P}$ are disjoint. Using the additivity of μ on $\mathscr{P}$, we obtain

$$\begin{aligned}\sum_{k=1}^{n} \mu(C_k) &= \sum_{k=1}^{n} \mu\left(\bigcup_{j=1}^{m} (C_k \cap D_j)\right) \\ &= \sum_{k=1}^{n}\sum_{j=1}^{m} \mu(C_k \cap D_j) = \sum_{j=1}^{m} \mu\left(\bigcup_{k=1}^{n} (C_k \cap D_j)\right) = \sum_{j=1}^{m} \mu(D_j).\end{aligned}$$

Note that the extension $\bar{\mu}$ is additive on $r(\mathscr{P})$.

II. Uniqueness of the extension. Let λ be an additive extension of the measure μ to $r(\mathscr{P})$. Then for every set $A \in r(\mathscr{P})$ we have an expression

$$A = \bigcup_{k=1}^{n} C_k, \quad \{C_1, \ldots, C_n\} \subset \mathscr{P}, \ C_k \cap C_j = \emptyset, \ k \ne j.$$

Consequently,

$$\lambda(A) = \sum_{k=1}^{n} \lambda(C_k) = \sum_{k=1}^{n} \mu(C_k) = \bar{\mu}(A).$$

III. σ-additivity of the extension. Let

$$\{A_n : \ n \ge 1\} \subset r(\mathscr{P}), \quad A_m \cap A_n = \emptyset, \ m \ne n; \quad A := \bigcup_{n=1}^{\infty} A_n \in r(\mathscr{P}).$$

Then

$$A = \bigcup_{j=1}^{m} B_j, \quad \{B_1, \ldots, B_m\} \subset \mathscr{P}, \quad B_k \cap B_j = \emptyset, \ k \ne j,$$

and for any $n \ge 1$

$$A_n = \bigcup_{k=1}^{r(n)} C_{nk}, \quad \{C_{n,1}, \ldots, C_{n,r(n)}\} \subset \mathscr{P}, \ C_{n,k} \cap C_{n,j} = \emptyset, \ k \ne j.$$

Using first the σ-additivity of μ on $\mathscr{P}$ and then the additivity of $\bar{\mu}$ on $r(\mathscr{P})$, we get

$$\begin{aligned}\bar{\mu} &= \sum_{j=1}^{m} \mu(B_j) = \sum_{j=1}^{m} \mu(B_j \cap A) = \sum_{j=1}^{m} \mu\left(B_j \cap \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{r(n)} C_{n,k}\right) \\ &= \sum_{j=1}^{m} \mu\left(\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{r(n)} (B_j \cap C_{n,k})\right) = \sum_{j=1}^{m}\sum_{n=1}^{\infty}\sum_{k=1}^{r(n)} \mu(B_j \cap C_{n,k}) = \sum_{n=1}^{\infty} \bar{\mu}(A_n).\end{aligned}$$

$\square$

3.2 Outer measure

Definition 3.2.1. A function $\lambda^* : 2^X \to (-\infty, +\infty]$ is called the **outer measure**, if

(i) $\lambda^*(\emptyset) = 0$ and λ^* is a nonnegative function

(ii) $\forall \{A,\, A_n,\, n \geq 1\} \subset 2^X,\ A \subset \bigcup_{n=1}^{\infty} A_n\!: \quad \lambda^*(A) \leq \sum_{n=1}^{\infty} \lambda^*(A_n).$

Exercise 3.2.2. Prove that an outer measure is monotone and semiadditive on 2^X.

Hint: For $A, B \in 2^X,\ A \subset B$ we have $A \subset B \cup \emptyset \cup \cdots \cup \emptyset \cup \ldots$

Definition 3.2.3. Let μ be a measure on a ring R of subsets of X. For every set $A \in 2^X$ we set

$$\mu^*(A) := \begin{cases} 0 & \text{if } A = \emptyset, \\ \inf\limits_{\{A_n:\, n\geq 1\}\subset R,\ A\subset \bigcup_{n=1}^{\infty} A_n} \sum\limits_{n=1}^{\infty} \mu(A_n) & \text{if there exists at least one such a sequence,} \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 3.2.4. *The function μ^* from Definition 3.2.3 is an outer measure.*

Proof. Condition (i) of Definition 3.2.1 is satisfied. We check Condition (ii). Let

$$\{A,\, A_n,\ n \geq 1\} \subset 2^X, \quad A \subset \bigcup_{n=1}^{\infty} A_n.$$

It is enough to consider the case where $\mu^*(A_n) < +\infty$, $n \geq 1$. According to Definition 3.2.3 and the definition of the infimum, we have

$$\forall \varepsilon > 0\ \ \forall n \geq 1\ \ \exists \big\{B_{n,j}:\ j \geq 1\big\} \subset R,\ \bigcup_{j=1}^{\infty} B_{n,j} \supset A_n :$$
$$\sum_{j=1}^{\infty} \mu(B_{n,j}) < \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

Hence, using the inclusion

$$\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} B_{n,j} \lim_{n\to\infty} \supset \bigcup_{n=1}^{\infty} A_n \supset A$$

and Definition 3.2.3, we obtain

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \mu(B_{n,j}) < \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

Making $\varepsilon \to 0+$, we get the following inequality

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

□

Remark 3.2.5. The function μ^* from Definition 3.2.3 is called the **outer measure generated by the measure μ**.

Exercise 3.2.6. Let $X = \mathbb{R}$, $\mathscr{P} = \{(k,k+1]: \ k \in \mathbb{Z}\} \cup \{\emptyset\}$ and

$$\lambda(\emptyset) := 0, \quad \lambda((k,k+1]) := 1, \ \ k \in \mathbb{Z}.$$

Prove that λ is a measure on $\mathscr{P}$. Let $\bar{\lambda}$ be the extension of λ to $r(\mathscr{P})$. Construct the outer measure λ^* generated by the measure $\bar{\lambda}$. Find $\lambda^*\left(\{\frac{1}{2}\}\right)$, $\lambda^*\left(\left(\frac{1}{2},\frac{3}{2}\right)\right)$, and $\lambda^*(\mathbb{N})$.

3.3 λ^*-measurable sets. Carathéodory theorem

Definition 3.3.1. Let λ^* be an outer measure on 2^X. A set $A \subset 2^X$ is called **λ^*-measurable**, if

$$\forall B \subset X: \ \ \lambda^*(B) = \lambda^*(B \cap A) + \lambda^*(B \setminus A).$$

Remark 3.3.2. 1. We note that $B \setminus A = B \cap A^c$ and $A^c = X \setminus A$.

2. For any sets $A, B \subset X$ we have $B = (B \cap A) \cup (B \setminus A)$, and, consequently,

$$\lambda^*(B) \leq \lambda^*(B \cap A) + \lambda^*(B \setminus A), \tag{3.3.1}$$

by the semiadditivity of the outer measure λ^*.

Exercise 3.3.3. Show that a set A is λ^*-measurable if and only if

$$\forall U \subset A \ \ \forall V \subset A^c: \ \ \lambda^*(U \cup V) = \lambda^*(U) + \lambda^*(V).$$

Exercise 3.3.4. Define a class of all λ^* measurable sets for the outer measure λ^* from Exercise 3.2.6.

Answer: It is the class consisting of at most countable union of sets from $\mathscr{P}$. The set $\left(\frac{1}{2},1\right]$ is not λ^*-measurable.

Theorem 3.3.5. *Let λ^* be an outer measure on 2^X and $\mathscr{S}$ be the class of all λ^*-measurable sets. Then the class $\mathscr{S}$ is a σ-algebra and the restriction of λ^* to $\mathscr{S}$ is a measure.*

Proof. I. $\mathscr{S}$ is an algebra. We note that $\emptyset \in \mathscr{S}$ because

$$\forall B \subset X: \ \ \lambda^*(B \cap \emptyset) + \lambda^*(B \setminus \emptyset) = \lambda^*(\emptyset) + \lambda^*(B) = \lambda^*(B).$$

Let $A \in \mathscr{S}$. Then $A^c \in \mathscr{S}$ also because

$$\forall B \subset X: \ \ \lambda^*(B \cap A^c) + \lambda^*(B \setminus A^c) = \lambda^*(B \cap A^c) + \lambda^*(B \cap A) = \lambda^*(B).$$

Take $G, F \in \mathscr{S}$. Then for every $B \subset X$ we have

$$\begin{aligned} \lambda^*(B) &= |\lambda^*\text{-measurability of } G| = \lambda^*(B \cap G) + \lambda^*(B \cap G^c) \\ &= |\lambda^*\text{-measurability of } F| = \lambda^*(B \cap G) + \lambda^*(B \cap G^c \cap F) + \lambda^*(B \cap G^c \cap F^c), \end{aligned} \tag{3.3.2}$$

$$\lambda^*(B\cap(G\cup F)) = |\lambda^*\text{-measurability of } G| = \lambda^*(B\cap(G\cup F)\cap G)+\lambda^*(B\cap(G\cup F)\cap G^c)$$
$$= \lambda^*(B\cap G)+\lambda^*(B\cap F\cap G^c). \tag{3.3.3}$$

By (3.3.2) and (3.3.3), we obtain the following equality

$$\lambda^*(B) = \lambda^*(B\cap(G\cup F))+\lambda^*(B\cap(G\cup F)^c).$$

Thus, $G\cup F\in\mathscr{S}$, and, consequently,

$$G\cap F = (G^c\cup F^c)^c\in\mathscr{S}, \quad G\setminus F = (G\cap F^c)\in\mathscr{S}.$$

II. $\mathscr{S}$ is a σ-algebra and the restriction of λ^* to $\mathscr{S}$ is a measure. Let $\{A_n:\ n\geq 1\}\subset\mathscr{S}$. We need to prove that $\bigcup_{n=1}^{\infty}A_n\in\mathscr{S}$. Since $\mathscr{S}$ is an algebra, without loss of generality we may assume that $A_m\cap A_n=\emptyset$, $m\neq n$. For every $B\subset X$ we have

$$\lambda^*(B\cap(A_1\cup A_2)) = \lambda^*(B\cap(A_1\cup A_2)\cap A_1)+\lambda^*(B\cap(A_1\cup A_2)\cap A_1^c)$$
$$= \lambda^*(B\cap A_1)+\lambda^*(B\cap A_2),$$

by the λ^*-measurability of A_1. The latter equality and the λ^*-measurability of A_3 yield

$$\lambda^*(B\cap(A_1\cup A_2\cup A_3)) = \lambda^*(B\cap A_3)+\lambda^*(B\cap(A_1\cup A_2)) = \sum_{k=1}^{3}\lambda^*(B\cap A_k).$$

Similarly, for each $n\geq 1$ we have the equality

$$\lambda^*\left(B\cap\bigcup_{k=1}^{n}A_k\right) = \sum_{k=1}^{n}\lambda^*(B\cap A_k). \tag{3.3.4}$$

Using now the λ^*-measurability of $\bigcup_{k=1}^{n}A_k$, equality (3.3.4) and the monotonicity of the outer measure, we obtain

$$\lambda^*(B) = \lambda^*\left(B\cap\bigcup_{k=1}^{n}A_k\right)+\lambda^*\left(B\cap\left(\bigcup_{k=1}^{n}A_k\right)^c\right)\geq\sum_{k=1}^{n}\lambda^*(B\cap A_k)+\lambda^*\left(B\cap\left(\bigcup_{k=1}^{n}A_k\right)^c\right).$$

Thus,

$$\lambda^*(B)\geq\sum_{k=1}^{\infty}\lambda^*(B\cap A_k)+\lambda^*\left(B\cap\left(\bigcup_{k=1}^{\infty}A_k\right)^c\right). \tag{3.3.5}$$

The latter inequality is based on Property (ii) of Definition 3.2.1. According to (3.3.1), we can conclude that

$$\lambda^*(B) = \lambda^*\left(B\cap\bigcup_{k=1}^{\infty}A_k\right)+\lambda^*\left(B\cap\left(\bigcup_{k=1}^{\infty}A_k\right)^c\right).$$

Hence, $\bigcup_{k=1}^{\infty}A_k\in\mathscr{S}$, and inequality (3.3.4) becomes the equality. Setting in (3.3.4) $B=\bigcup_{k=1}^{\infty}A_k$, we get

$$\lambda^*\left(\bigcup_{k=1}^{\infty}A_k\right) = \sum_{k=1}^{\infty}\lambda^*(A_k).$$

□

Exercise 3.3.6. Give an example of outer measure λ^* on 2^X such that $\mathscr{S} = \{\emptyset, X\}$.

3.4 Complete measures

Definition 3.4.1. Let μ be a measure on a σ-algebra $\mathscr{S}$. The measure μ is called **complete**, if

$$\forall A \in \mathscr{S},\ \mu(A) = 0 \quad \forall B \subset A:\ B \in \mathscr{S}.$$

Remark 3.4.2. If $A \in \mathscr{S}$, $\mu(A) = 0$, $B \subset A$ and $B \in \mathscr{S}$, then $\mu(B) = 0$, by the monotonicity of measure.

Corollary 3.4.3. *Under the conditions of Theorem 3.3.5, the measure λ^* is complete of $\mathscr{S}$.*

Proof. Let $A \in \mathscr{S}$, $\lambda^*(A) = 0$ and $C \subset A$. By the monotonicity of the outer measure λ^* and the λ^*-measureability of A, we have that for every $B \subset X$

$$\lambda^*(B) \geq \lambda^*(B \cap C^c) \geq \lambda^*(B \cap A^c) = \lambda^*(B \cap A) + \lambda^*(B \cap A^c) = \lambda^*(B),$$

since $0 \leq \lambda^*(B \cap A) \leq \lambda^*(A) = 0$. Similarly, we can obtain the equality $\lambda^*(B \cap C) = 0$. Hence, $C \in \mathscr{S}$. $\square$

Exercise 3.4.4. Let μ be a measure on a σ-algebra $\mathscr{S}$, and

$$\mathscr{S}^0 = \{A \cup \Phi :\ A \in \mathscr{S},\ \exists B \in \mathscr{S},\ \mu(B) = 0,\ \Phi \subset B\}, \quad \mu^0(A \cup \Phi) := \mu(A),\ A \cup \Phi \in \mathscr{S}^0.$$

Prove that $\mathscr{S}^0$ is a σ-algebra and μ^0 is a complete measure on $\mathscr{S}^0$.

3.5 Measurability of sets of the initial ring

If λ^* is a measure, then the class $\mathscr{S}$ of all λ^*-measurable sets is a σ-algebra, according to Theorem 3.3.5. However this σ-algebra can be very poor. It is possible that $\mathscr{S} = \{\emptyset, X\}$.

We now consider the case, where the outer measure μ^* is generated by a measure μ defined on a ring R. As above, $\mathscr{S}$ will be the class of all μ^*-measurable subsets of X. Denote also

$$\bar{\mu}(A) := \mu^*(A), \quad A \in \mathscr{S}.$$

The measure $\bar{\mu}$ is the extension of the measure μ from the ring R to the σ-algebra $\mathscr{S}$ if $R \subset \mathscr{S}$.

Theorem 3.5.1. *$R \subset \mathscr{S}$ and the measure $\bar{\mu}$ is the extension of the measure μ from the ring R to the σ-algebra $\mathscr{S}$.*

Proof. I. We first prove that

$$\forall A \in R: \quad \mu^*(A) = \mu(A).$$

Indeed, $\mu^*(A) \le \mu(A)$ since $A \subset A \cup \emptyset \cup \emptyset \cup \ldots$. Moreover, for each sequence $\{A_n: \ n \ge 1\} \subset R$, $A \subset \bigcup_{n=1}^{\infty} A_n$ we have $A = \bigcup_{n=1}^{\infty} (A \cap A_n)$. The σ-additivity and the monotonicity of the measure μ on R yield the inequality

$$\mu(A) \le \sum_{n=1}^{\infty} \mu(A \cap A_n) \le \sum_{n=1}^{\infty} \mu(A_n).$$

Thus, according to Definition 3.2.3, $\mu(A) \le \mu^*(A)$.

II. $R \subset \mathscr{S}$. Let $A \in R$ and $\varepsilon > 0$ be fixed. We consider an arbitrary set $B \subset X$, $\mu^*(B) < +\infty$. According to Definition 3.2.3

$$\exists \{A_n: \ n \ge 1\} \subset R: \quad \mu^*(B) + \varepsilon > \sum_{n=1}^{\infty} \mu(A_n).$$

Hence, by the additivity of the measure μ on R and Definition 3.2.3, we get

$$\mu^*(B) + \varepsilon > \sum_{n=1}^{\infty} (\mu(A_n \cap A) + \mu(A_n \cap A^c)) \ge \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Making now $\varepsilon \to 0+$,

$$\mu^*(B) \ge \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

This inequality and the simiadditivity of outer measure (3.3.1) implies the μ^*-measurability of the set A. □

Exercise 3.5.2. Check that $\sigma r(R) \subset \sigma a(R) \subset \mathscr{S}$.

Exercise 3.5.3. Let μ be a σ-finite measure on a ring R. Then the outer measure μ^* on 2^X and the measure $\bar{\mu}$ on $\mathscr{S}$ are σ-finite.

Exercise 3.5.4.* For $A \in 2^X$ we set

$$\mu^{**} := \inf \left\{ \sum_{n=1}^{\infty} \bar{\mu}(A_n): \ \{A_n: \ n \ge 1\} \subset \mathscr{S}, \ \bigcup_{n=1}^{\infty} A_n \supset A \right\}.$$

Prove that $\mu^{**} = \mu^*$.

3.6 Uniqueness of extension

Let $\bar{\mu}$ be the extension of a measure μ from a ring R to the σ-algebra $\mathscr{S}$ of all μ^*-measurable sets. Since $\mathscr{S}$ is a σ-algebra and $R \subset \mathscr{S}$, we have that $\sigma r(R) \subset \mathscr{S}$.

Theorem 3.6.1. *The extension of σ-finite measure μ from a ring R to $\sigma r(R)$ is unique and σ-finite.*

Proof. Let a measure λ be an extension of μ to $\sigma r(R)$. We first assume that λ and $\bar{\mu}$ are finite of $\sigma r(R)$. Set

$$Q := \{A \in \sigma r(R) : \ \lambda(A) = \bar{\mu}(A)\}.$$

Then $R \subset Q \subset \sigma r(R)$. The family of sets Q is a monotone class. Indeed, for a sequence

$$\{A_n : \ n \geq 1\} \subset Q \quad A_n \subset A_{n+1}, \ \ n \geq 1,$$

we have

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \lambda(A_n) = \lim_{n\to\infty} \bar{\mu}(A_n) = \bar{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right),$$

by Theorem 2.3.1. Hence $\bigcup_{n=1}^{\infty} A_n \in Q$. Similarly, using the assumption of finiteness of one of the measures λ, $\bar{\mu}$ and Theorem 2.3.1, one can check that the limit of a decreasing sequence of sets from Q also belongs to Q.

Thus, $m(R) \subset Q \subset \sigma r(R)$. Moreover, $m(R) = \sigma r(R)$, according to Theorem 1.4.6.

II. Let $A \in R$ be a set such that $\lambda(A)$ or $\bar{\mu}(A)$ is finite. Then according to Part I. of the proof, the measures λ and $\bar{\mu}$ coincide on $A \cap \sigma r(R) = \sigma r(A \cap R)$. Moreover, each set from $\sigma r(R)$ is contained in an union of countable number of sets from R which have a finite measure $\bar{\mu}$. □

Exercise 3.6.2.* Prove that the measure $\bar{\mu}$ on $\mathscr{S}$ is the complement of the measure $\bar{\mu}$ considered on $\sigma r(R)$.

Remark 3.6.3. The condition of σ-finiteness of the measure μ on R in Theorem 3.6.1 is essential. See, e.g. The example in [Hal50, Section 3.13].

3.7 Approximation theorem

Theorem 3.7.1. *Let μ be a σ-finite measure on a ring R and $\bar{\mu}$ be its extension to $\sigma r(R)$. Then*

$$\forall A \in \sigma r(R), \ \ \bar{\mu}(A) < +\infty \ \ \forall \varepsilon > 0 \ \ \exists C \in R : \ \ \bar{\mu}((A \setminus C) \cup (C \setminus A)) < \varepsilon.$$

Proof. Let μ^* be the outer measure generated by the measure μ. Remark that $\bar{\mu} = \mu^*$ on $\sigma r(R)$. Let $\varepsilon > 0$ be fixed. According to the definition of the outer measure for $\bar{\mu}(A) = \mu^*(A)$ and the number $\frac{\varepsilon}{2}$, we have

$$\exists \{A_n : \ n \geq 1\} \subset R, \ \ \bigcup_{n=1}^{\infty} A_n \supset A : \ \ \bar{\mu}(A) + \frac{\varepsilon}{2} > \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \bar{\mu}(A_n).$$

Hence, using the σ-additivity and the monotonicity of the measure $\bar{\mu}$, we obtain for every $n \geq 1$

$$\bar{\mu}(A) + \frac{\varepsilon}{2} > \bar{\mu}\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \bar{\mu}\left(\bigcup_{k=1}^{n} A_k\right). \tag{3.7.1}$$

Moreover, by the continuity of the measure $\bar{\mu}$ from below,

$$\bar{\mu}\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n\to\infty} \bar{\mu}\left(\bigcup_{k=1}^{n} A_k\right).$$

Therefore,

$$\exists n_0 \geq 1: \quad \bar{\mu}\left(\bigcup_{k=1}^{n_0} A_k\right) + \frac{\varepsilon}{2} > \bar{\mu}\left(\bigcup_{k=1}^{\infty} A_k\right). \tag{3.7.2}$$

Let $C := \bigcup_{k=1}^{n_0} A_k$. Then inequalities (3.7.1) and (3.7.2) yield

$$\bar{\mu}(C \setminus A) \leq \bar{\mu}\left(\left(\bigcup_{k=1}^{\infty} A_k\right) \setminus A\right) < \frac{\varepsilon}{2},$$
$$\bar{\mu}(A \setminus C) \leq \bar{\mu}\left(\left(\bigcup_{k=1}^{\infty} A_k\right) \setminus \left(\bigcup_{k=1}^{n_0} A_k\right)\right) < \frac{\varepsilon}{2}.$$

□

Exercise 3.7.2. Let μ^* be the outer measure generated by a measure μ defined on a ring R and $\mu^*(X) < +\infty$. Prove that

$$A \in \mathscr{S} \iff \forall \varepsilon > 0 \ \exists C \in R: \ \mu^*((A \setminus C) \cup (C \setminus A)) < \varepsilon.$$

3.8 Lebesgue measure on the real line

Let $X = \mathbb{R}$, $\mathscr{P}_1 = \{(a,b]: \ -\infty < a < b < +\infty\} \cup \{\emptyset\}$ be a semiring of subsets of $\mathbb{R}$ and

$$\lambda(\emptyset) := 0, \quad \lambda((a,b]) := b - a, \ (a,b] \in \mathscr{P}_1.$$

The values of λ at a set from $\mathscr{P}_1$ is the length of this set. According to Corollary 2.4.2, the function λ is a measure on $\mathscr{P}_1$. By Theorem 3.1.2, the function λ can be uniquely extended to a measure on $r(\mathscr{P}_1)$. We denote this extension also by λ. Let λ^* be the outer measure generated by λ and defined on all subsets of $\mathbb{R}$, and $\mathscr{S}$ be the class of all λ^*-measurable sets. According to Theorem 3.3.5, the class $\mathscr{S}$ is a σ-algebra and λ^* is a measure on $\mathscr{S}$.

Definition 3.8.1. Sets from the σ-algebra $\mathscr{S}$ are said to be **Lebesgue measurable sets** and the measure λ^* on $\mathscr{S}$ (further denoting by λ) is called the **Lebesgue measure** (or **one-dimensional Lebesgue measure**).

The following inclusions

$$\mathscr{P}_1 \subset r(\mathscr{P}_1) \subset \mathscr{B}(\mathbb{R}) \subset \mathscr{S}$$

hold. Thus, all Borel sets are Lebesgue measurable. Let us consider some examples of Borel sets on $\mathbb{R}$ and compute their Lebesgue measure.

Example 3.8.2. For every $x \in \mathbb{R}$ the one point set $\{x\}$ belongs to $\mathscr{B}(\mathbb{R})$ because

$$\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x \right].$$

According to Theorem 2.3.3 and the fact that $\lambda((x-1,x]) = 1$, we can conclude that

$$\lambda(\{x\}) = \lim_{n\to\infty} \lambda \left(\bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x \right] \right) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, any countable set $A \leq \mathbb{R}$ is a Borel set and its Lebesgue measure equals 0, according to the σ-additivity of λ. In particular, $\lambda(\mathbb{Q}) = 0$. Moreover,

$$\lambda([a,b]) = \lambda(\{a\}) + \lambda((a,b]) = 0 + (b-a) = b-a, \quad \lambda((a,b)) = b-a.$$

Example 3.8.3. Let $G \neq \emptyset$ be an open subset of $\mathbb{R}$. Then G is a Borel set. Then

$$G = \bigcup_{k\geq 1} (\alpha_k, \beta_k), \quad (\alpha_j, \beta_j) \cap (\alpha_k, \beta_k) = \emptyset, \;\; j \neq k, \;\; \alpha_k < \beta_k, \;\; k \geq 1.$$

According to the σ-additivity of the Lebesgue measure, we obtain

$$\lambda(G) = \sum_{k\geq 1} \lambda((\alpha_k, \beta_k)) = \sum_{k\geq 1} (\beta_k - \alpha_k).$$

Remark 3.8.4. Using the last equality as a definition, E. Borel constructed the extension of the length to $\mathscr{B}(\mathbb{R})$ in 1898.

Exercise 3.8.5. Prove that the length (the Borel measure) on the σ-algebra of Borel sets $\mathscr{B}(\mathbb{R})$ is an incomplete measure.

Exercise 3.8.6. Check that the Borel set $\mathbb{Q} \cap [0,1]$ is not Jordan measurable.

Exercise 3.8.7.* **The Cantor set**. Let

$$J_0 := [0,1], \;\; J_1 := [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3} \right) = \left[0, \frac{1}{3} \right] \cup \left[\frac{2}{3}, 1 \right],$$
$$J_2 := \frac{1}{3} J_1 \cup \left(\frac{2}{3} + \frac{1}{3} J_1 \right), \;\; \ldots, \;\; J_n := \frac{1}{3} J_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3} J_{n-1} \right), \;\; \ldots$$

The set J_n is the union of 2^n intervals of the length 3^{-n} each, $n \geq 1$. Let

$$U := \bigcap_{n=1}^{\infty} J_n.$$

Prove that the closed set U has the cardinality of the continuum and is Jordan measurable. Prove that all subsets of U also Jordan measurable. Show that the class $\mathscr{B}(\mathbb{R})$ has the cardinality of the continuum. Consequently, there exist Jordan measurable sets that are not Borel sets.

Exercise 3.8.8.* Represent the interval $[0,1)$ as the union of disjoint sets as follows. Two numbers x and y from $[0,1)$ belongs the same set if and only if their difference $x-y$ is a rational number. From every set we take a number z. Let A denote the set of all such z. Show that the set A is not Lebesgue measurable.

Hint: For $r\in[0,1)\cap\mathbb{Q}$ set

$$A_r := \{x+r:\ x\in A,\ x+r<1\}\cup\{x+r-1:\ x\in A,\ x+r>1\}.$$

Prove that $A_r\cap A_{r'}=\emptyset$, $r\neq r'$, $\lambda^*(A_r)=\lambda^*(A_{r'})$, $A=\bigcup_{r\in[0,1)\cup\mathbb{Q}}A_r$.

Exercise 3.8.9.* Let $\mathscr{K}$ be the ring of all Jordan measurable sets on $\mathbb{R}$. Prove that $\mathscr{K}\subset\mathscr{S}$.

Exercise 3.8.10. Let $\{A_n:\ n\geq 1\}$ be a decreasing sequence of open subsets of $[0,1]$ such that $\inf\limits_{n\geq 1}\lambda(A_n)>0$. Prove that $\bigcup_{n=1}^{\infty}A_n\neq\emptyset$.

Exercise 3.8.11. Leg $A\in\mathscr{S}$, $a\in\mathbb{R}$, and $B:=a+A:=\{a+x:\ x\in A\}$. Prove that $B\in\mathscr{S}$ and $\lambda(A)=\lambda(B)$.

Exercise 3.8.12. Show that $\lambda(A)<+\infty$ for every bounded set $A\in\mathscr{B}(\mathbb{R})$.

Exercise 3.8.13.* Let A_0 (resp. A_f) be the set of all numbers of the interval $[0,1]$ whose decimal representations do not contain the digit 3 (resp. decimal representations contain the digit 3 only a finite number of times). Prove that

$$A_0, A_f\in\mathscr{S},\quad \lambda(A_0)=\lambda(A_f).$$

Exercise 3.8.14. Let $A\in\mathscr{S}$, $\lambda(A)<+\infty$ and $f(x):=\lambda(A\cap(-\infty,x))$, $x\in\mathbb{R}$. Prove that $f\in\mathrm{C}(\mathbb{R})$.

Exercise 3.8.15. Let $A\in\mathscr{S}$ be a bounded set with $\lambda(A)>0$. Prove that

$$\forall\alpha\in(0,\lambda(A))\ \ \exists B\subset A,\ \ B\in\mathscr{S}:\ \ \lambda(B)=\alpha.$$

Exercise 3.8.16.* Prove the existence of a set $A\in\mathscr{S}$, $\lambda(A)>0$, which contains no intervals (α,β), $\alpha<\beta$.

3.9 Lebesgue measure on $\mathbb{R}^m$

Let $X=\mathbb{R}^d$, $d\in\mathbb{N}$,

$$\mathscr{P}_d := \left\{\prod_{k=1}^{d}(a_k,b_k]:\ -\infty<a_k<b_k<+\infty,\ 1\leq k\leq m\right\}\cup\{\emptyset\}$$

be a semiring of subsets of $\mathbb{R}^d$ and

$$\lambda_d(\emptyset):=0,\quad \lambda_d:=\left(\prod_{k=1}^{d}(a_k,b_k]\right)=\prod_{k=1}^{d}(b_k-a_k).$$

The case $d=1$ was considered in Section 3.8. For $d=2$ the function λ_2 is the area of rectangles $(a_1,b_1]\times(a_2,b_2]$ from $\mathscr{P}_2$. For $d=3$ the function λ_3 is the volume of boxes $(a_1,b_1]\times(a_2,b_2]\times(a_3,b_3]$ from $\mathscr{P}_3$.

By Theorem 2.4.1, the function λ_d is a measure on the semiring $\mathscr{P}_d$. This measure can be uniquely extended to the measure on $r(\mathscr{P}_d)$, denoted also by λ_d, according to Theorem 3.1.2. Let λ_d^* be the outer measure generated by λ_d and $\mathscr{S}_d$ be the class of all λ_d^*-measurable subsets of $\mathbb{R}^d$. By the Caratheodory Theorem 3.3.5, the class $\mathscr{S}_d$ is a σ-algebra and λ_d^* is a measure on $\mathscr{S}_d$.

Definition 3.9.1. Sets from the σ-algebra $\mathscr{S}$ are said to be **Lebesgue measureable sets** and the measure λ_d^* on $\mathscr{S}$ (furthere denoting by λ_d) is called the **Lebesgue measure** (or **d-dimensional Lebesgue measure**).

The following inclusions

$$\mathscr{P}_d \subset r(\mathscr{P}_d) \subset \mathscr{B}(\mathbb{R}^d) \subset \mathscr{S}_d$$

hold.

Exercise 3.9.2. Prove the following statements:

a) for every poing $(x,y)\in\mathbb{R}^2$ the one-point set $\{(x,y)\}\in\mathscr{S}_2$ and $\lambda_2(\{(x,y)\})=0$;

b) a line segment $I=\{(x,y):\ a\le x\le b,\ y=c\}$ belongs to $\mathscr{S}_2$ and $\lambda_2(I)=0$;

c) if l is a line in $\mathbb{R}^2$, then $l\in\mathscr{S}_2$ and $\lambda_2(l)=0$;

d) the set $\Gamma=\{(x,f(x)):\ x\in\mathbb{R}\}$ belongs to $\mathscr{S}_2$ and $\lambda_2(\Gamma)=0$, where $f\in \mathrm{C}(\mathbb{R})$;

e) the set $F=\{(x,y):\ a\le x\le b,\ 0\le y\le f(x)\}$ belongs to $\mathscr{S}_2$ and $\lambda_2(F)=\int_a^b f(x)dx$, where $f\in\mathrm{C}([a,b])$, $f(x)\ge 0$, $x\in[a,b]$.

Exercise 3.9.3. Let

$$X=\mathbb{R}^2,\quad \mathscr{P}=\{(a,b]\times\mathbb{R}:\ -\infty<a<b<+\infty\}\cup\{\emptyset\},\quad \mu(\emptyset):=0,\ \ \mu((a,b]\times\mathbb{R}):=b-a.$$

Prove that $\mathscr{P}$ is a semiring and μ is a measure on $\mathscr{P}$. Let μ^* be the outer measure generated by μ. Find $\mu^*(A)$ for the following sets:

a) $A=\{(x,y):\ 0\le x\le 1,\ y=x\}$; b) $A=\left\{(x,y):\ x^2+y^2\le 1\right\}$.

Describe the class of all μ^*-measurable functions.

Exercise 3.9.4. Let $X=\mathbb{R}^2$, $\mathscr{S}=\mathscr{S}_2$ and λ_2 be the Lebesgue measure on $\mathscr{S}_2$. We set

$$f(t):=\lambda_2\left(A\cap\left\{(x,y):\ x^2+y^2\le t^2\right\}\right),\quad t\ge 0,$$

for a set $A\in\mathscr{S}_2$. Prove that $f\in\mathrm{C}([0,+\infty))$.

Exercise 3.9.5. Let the conditions of the previous exercise are satisfied and, additionally, $\lambda_2(A) < +\infty$. Set

$$f(t) := \lambda_2\left(A \cap \left\{(x,y) : \; tx - y^2 \le 0\right\}\right), \quad t > 0.$$

Prove that $f \in \mathrm{C}((0,+\infty))$.

Exercise 3.9.6. Prove that a set $A \subset \mathbb{R}^d$ is Lebesgue measurable if and only if

$$\forall \varepsilon > 0 \;\; \exists G \text{ open}, \;\; G \supset A : \;\; \lambda_d^*(G \setminus A) < \varepsilon.$$

3.10 Lebesgue-Stieltjes measure on the real line

Let $X = \mathbb{R}$, $\mathscr{P}_1 = \{(a,b] : \; -\infty < a < b < +\infty\} \cup \{\emptyset\}$ and $F : \mathbb{R} \to \mathbb{R}$ be a nondecreasing right continuous function. We set

$$\lambda_F(\emptyset) := 0, \quad \lambda_F((a,b]) := F(b) - F(a), \;\; (a,b] \in \mathscr{P}_1.$$

According to Theorem 2.4.4, the function λ_F is a measure on $\mathscr{P}_1$. By Theorem 3.1.2, there exists a unique extension of this measure to $r(\mathscr{P}_1)$, which we will also denote by λ_F. Let λ_F^* be the outer measure generated by λ_F and $\mathscr{S}_F$ be the class of all λ_F^*-measurable subsets of $\mathbb{R}$. By Caratheodory Theorem 3.3.5, the class $\mathscr{S}_F$ is a σ-algebra and λ_F^* is a measure on $\mathscr{S}_F$.

Definition 3.10.1. Sets from the σ-algebra $\mathscr{S}_F$ are said to be **measurable** and the measure λ_F^* on $\mathscr{S}_F$ (further denoted by λ_F) is called the **Lebesgue-Stieltjes measure**.

The following inclusion

$$\mathscr{P}_1 \subset r(\mathscr{P}_1) \subset \mathscr{B}(\mathbb{R}) \subset \mathscr{S}_F$$

hold.

Exercise 3.10.2. Prove that

$$\forall x \in \mathbb{R} : \;\; \{x\} \in \mathscr{S}_F \;\text{ and }\; \lambda_F(\{x\}) = F(x) - F(x-).$$

Exercise 3.10.3. Show that that there exists at most countable set $J \subset \mathbb{R}$ such that

$$\forall x \in J : \;\; \lambda_F(\{x\}) > 0.$$

Exercise 3.10.4. Prove that the measure λ_F coincides with the Lebesgue measure λ if $F(x) = x$, $x \in \mathbb{R}$.

Remark 3.10.5. A similar definition to Definition 3.10.1 can be done for the Lebesgue-Stieltjes measure on $\mathbb{R}^d$.

3.11 Some general definitions

Definition 3.11.1. Let X be a fundamental set and $\mathscr{F}$ be a σ-algebra of subsets of X. The pair $(X,\mathscr{F})$ is called a **measurable space**, and sets from $\mathscr{F}$ are said to be **measurable**. Let μ be a measure on $\mathscr{F}$. The triple $(X,\mathscr{F},\mu)$ is called a **space with measure**. If $\mu(X)=1$, then μ is called a **probability measure**, and the space $(X,\mathscr{F},\mu)$ is called a **probability space**.

The introduced terminology is common and we will use it hereinafter.

Chapter 4

Measurable maps and functions

4.1 Definition and examples

We first remind some facts related with the notion of map $T : X \to X'$, where X and X' are some sets.

The **image** of a set $A \subset X$ under the map T is the set $TA := \{Tx : \ x \in A\}, \ T\emptyset := \emptyset$.

The **preimage** of a set $A' \subset X'$ under the map T is the set $T^{-1}A' := \{x : \ Tx \in A'\}, \ T^{-1}\emptyset := \emptyset$.

Further elementary properties of images and preimages are listed in the exercises below. There ω denotes an arbitrary set of indices.

Exercise 4.1.1. Let $A_\alpha \subset X$ for every $\alpha \in \omega$. Prove that

$$T\left(\bigcup_{\alpha\in\omega} A_\alpha\right) = \bigcup_{\alpha\in\omega} TA_\alpha.$$

Exercise 4.1.2. Let $A'_\alpha \subset X'$ for every $\alpha \in \omega$. Prove the following equalities

$$T^{-1}\left(\bigcup_{\alpha\in\omega} A'_\alpha\right) = \bigcup_{\alpha\in\omega} T^{-1}A'_\alpha, \quad T^{-1}\left(\bigcap_{\alpha\in\omega} A'_\alpha\right) = \bigcap_{\alpha\in\omega} T^{-1}A'_\alpha,$$
$$T^{-1}\left(A'_{\alpha_1} \setminus A'_{\alpha_2}\right) = \left(T^{-1}A'_{\alpha_1}\right) \setminus \left(T^{-1}A'_{\alpha_2}\right).$$

Exercise 4.1.3. Let $\mathscr{F}'$ be a σ algebra of subsets of X'. Prove that the class of sets

$$T^{-1}\mathscr{F}' := \left\{T^{-1}A' : \ A' \in \mathscr{F}'\right\}$$

is a σ-algebra of subsets of X.

Exercise 4.1.4. Let $X = X' = \mathbb{R}$, $\mathscr{F}' = \mathscr{B}(\mathbb{R})$. Find $T^{-1}\mathscr{F}'$ in the following cases:

(i) $Tx = x^3$, (ii) $Tx = |x|$, (iii) $Tx = \begin{cases} 0, & \text{if } x \le 0, \\ x, & \text{if } x > 0, \end{cases}$, (iv) $Tx = \cos x$; $x \in \mathbb{R}$.

Definition 4.1.5. Let $(X,\mathscr{F})$, $(X',\mathscr{F}')$ be measurable spaces and $f: X \to X'$. The map f is called $(\mathscr{F},\mathscr{F}')$**-measurable**, if $f^{-1}\mathscr{F}' \subset \mathscr{F}$, that is,

$$\forall A' \in \mathscr{F}': \;\; f^{-1}(A') \in \mathscr{F}.$$

The map f is said to be $\mathscr{F}$**-measurable** in the case $X' = \mathbb{R}$, $\mathscr{F}' = \mathscr{B}(\mathbb{R})$.

Exercise 4.1.6. Let $\mathscr{F} = 2^X$. Prove that any map $f: X \to X'$ is $(\mathscr{F},\mathscr{F}')$-measurable.

Exercise 4.1.7. Let $\mathscr{F} = \{\emptyset, X\}$. Which functions are $\mathscr{F}$-measurable?

Chapter 5

Appendix

5.1 Structure of σ-algebras

Here we discuss the equivalence which is defined on the universal set X by the σ-algebra of its subsets. This will lead to the description of the finite σ-algebra and can be a starting point for studying of conditional measures.

Exercise 5.1.1. Let M be a class of subsets of X. We will say that $x \sim_M y$ if and only if there exists no such $A \in M$ that only one from x, y belongs to A. Prove that $\sim_M$ is an equivalence relation on X.

Exercise 5.1.2. Suppose that M is finite. Prove that all equivalence classes with respect to $\sim_M$ can be expressed as $\bigcap_{A \in M} A^{\varepsilon}$, where $\varepsilon = \pm 1$ and $A^1 := A$, $A^{-1} := A^c = X \setminus A$.

Exercise 5.1.3. Assume that M is finite σ-algebra. Prove that all equivalence classes with respect to $\sim_M$ belongs to M.

Let us denote by $H_1, \ldots, H_n$ the equivalence classes from the previous exercise.

Exercise 5.1.4. Check that under condition of the Exercise 5.1.3 every element of M is a union of certain elements from $H_1, \ldots, H_m$.

Exercise 5.1.5. Prove that for any finite σ-algebra M there exists a natural number n such that the number of sets in M equals 2^n.

Exercise 5.1.6. Let X be the Euclidean space $\mathbb{R}^d$ and $\mathscr{B}(\mathbb{R}^d)$ be a Borel σ-algebra in $\mathbb{R}^d$. Prove that equivalence classes for $\sim_{\mathscr{B}(\mathbb{R}^d)}$ are one-point sets.

Let f be a function from X to Y and $\mathscr{A}$ be a σ-algebra of subsets in Y.

Exercise 5.1.7. Check that the family

$$\Gamma = \left\{ f^{-1}(A) : A \in \mathscr{A} \right\}$$

is a σ-algebra of subsets in X.

Exercise 5.1.8. Prove that equivalence classes for $\sim_\Gamma$ can be described as $f^{-1}(Z)$, where Z are equivalence classes for $\sim_{\mathscr{A}}$.

Exercise 5.1.9. * Give an example of a set X and a σ-algebra M of its subsets such that the equivalence classes with respect to $\sim_M$ do not belong to M.

Bibliography

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