Problem sheet 6

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1061024
until 22:00, May 27.

Let $(X, \mathcal{F})$ denote a measurable space and λ be a measure on $\mathcal{F}$. All functions considered here are $\mathcal{F}$-measurable.

1. (i) **[3 points]** Let $A \in \mathcal{F}$ and for every $n \geq 1$, functions $f_n \in L(A, \lambda)$ and are non-negative. Let also the series $\sum_{n=1}^{\infty} f_n$ converges λ-a.e. on A.[1] Show that
$$\sum_{n=1}^{\infty} \int_A f_n d\lambda = \int_A \left(\sum_{n=1}^{\infty} f_n \right) d\lambda.$$
(ii) **[3 bonus points]** Let $X = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, λ be a Lebesgue measure on $[0,1]$. Consider the integrable functions $f_1 = 2\mathbb{I}_{[0,\frac{1}{2}]}$, $\quad f_n = \mathbb{I}_{[0,\frac{1}{n+1}]} - n\mathbb{I}_{(\frac{1}{n+1},\frac{1}{n}]}$, $\quad n \geq 2$. Show that the series $\sum_{n=1}^{\infty} f_n$ converges a.e. on $[0,1]$ but
$$\sum_{n=1}^{\infty} \int_0^1 f_n dx \neq \int_0^1 \left(\sum_{n=1}^{\infty} f_n \right) dx.$$

2. **[3 points]** Let p_k, $k \geq 1$, be a non-negative numbers satisfying the condition
$$\sup_{0<s<1} \sum_{k=1}^{\infty} \frac{\sin^2(sk)}{s^2} p_k < +\infty.$$
Prove that $\sum_{k=1}^{\infty} k^2 p_k < +\infty$.
Hint: Set $X = \mathbb{N}$, $\lambda(\{k\}) = p_k$, $k \in \mathbb{N}$, and use Fatou's lemma.

3. **[4 points]** Let $f : [0,1] \to \mathbb{R}$ be non-negative and Borel measurable. Compute
$$\lim_{n \to \infty} \int_0^1 x^{nf(x)} dx.$$

4. **[3 points]** Compute the sum $\sum_{n=1}^{\infty} \int_1^{+\infty} \frac{dx}{(1+x^2)^n}$.

5. **[4 bonus points]** Let F be a continuously differentiable non-decreasing function on $\mathbb{R}$ with $F' = f$. Show that
$$\int_A g(x) dF(x) = \int_A g(x) f(x) dx$$
for every non-negative function g on $\mathbb{R}$ and a Borel set A.

6. **[2 points]** Let $l_n^{\infty} := \mathbb{R}^n$ and $d(x,y) = \max_{k=1,\dots,n} |\xi_k - \eta_k|$, $x = (\xi_k)_{k=1}^n$, $y = (\eta_k)_{k=1}^n \in l_n^{\infty}$. Show that (l_n^{∞}, d) is a metric space.

7. **[2 points]** Draw the balls $B_1(0)$ in the following metric spaces l_2^1, l_2^2 and l_2^{∞}.

8. **[2+2 points]** Compute the distances between the functions $\cos x$ and $\sin x$, $x \in [0, 2\pi]$, in a) $C[0, 2\pi]$ and b) $L_2[0, 2\pi]$.

[1] The series $\sum_{n=1}^{\infty} f_n$ converges λ-a.e. on A if the sequence of functions $S_n = \sum_{k=1}^{n} f_k$ converges λ-a.e. on A