Problem sheet 4

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1035072
until 22:00, May 13.

Let $(X, \mathcal{F})$ denote a measurable space.

1. **[3 points]** Let X, X' be sets, and $\mathcal{F}'$ be a σ-algebra on X'. Let also $f : X \to X'$ be a function. Show that the class of sets
$$f^{-1}(\mathcal{F}') := \left\{ f^{-1}(A') : \ A' \in \mathcal{F}' \right\}$$
is a σ-algebra on X.

2. **[2 points]** Prove that every Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ is also Lebesgue measurable.[1]

3. **[4 points]** Let $f_k : X \to \mathbb{R}$, $k = 1, \dots, m$, be $\mathcal{F}$-measurable functions. Consider $f(x) := (f_1(x), \dots, f_m(x))$, $x \in X$. Show that the function $f : X \to \mathbb{R}^m$ is also $\mathcal{F}$-measurable, that is, $f^{-1}(A') \in \mathcal{F}$ for every $A' \in \mathcal{B}(\mathbb{R}^m)$.

4. **[3 points]** Let $f, g : X \to \mathbb{R}$ be $\mathcal{F}$-measurable. Show that $\{x \in X : \ f(x) \geq g(x)\}$ and $\{x \in X : f(x) = g(x)\}$ belong to $\mathcal{F}$.

5. **[3 points]** Let for a function $f : \mathbb{R} \to \mathbb{R}$ there exists the derivative f' on $\mathbb{R}$. Prove that f' is a Borel function.

6. **[3 points]** Let $f_1, f_2 : X \to \mathbb{R}$ be non-negative simple functions such that $f_1(x) \leq f_2(x)$, $x \in X$. Using the definition of the Lebesgue integral show that
$$\int_A f_1 d\lambda \leq \int_A f_2 d\lambda,$$
for all $A \in \mathcal{F}$.

7. **[2 bonus points]** Let functions $f, g : \mathbb{N} \to \mathbb{R}$ satisfy
$$f^{-1}(\mathcal{B}(\mathbb{R})) \subset g^{-1}(\mathcal{B}(\mathbb{R})).$$
Show that there exists a Borel function $F : \mathbb{R} \to \mathbb{R}$ such that $f(x) = F(g(x))$, $x \in \mathbb{N}$.

8. **[2+2 points]** Let $X = \mathbb{N}$, $\mathcal{F} = 2^{\mathbb{N}}$ and $\lambda(\emptyset) := 0$, $\lambda(A) := \sum_{n \in A} \frac{1}{n}$, $A \in 2^{\mathbb{N}}$. Show that
   a) $f \in L(\mathbb{N}, \lambda)$ if and only if $\sum_n^\infty \frac{|f(n)|}{n} < +\infty$;
   b) $\int_{\mathbb{N}} f d\lambda = \sum_{n=1}^\infty \frac{f(n)}{n}$ for $f \in L(\mathbb{N}, \lambda)$.

[1] A function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if it is $\mathcal{S}$-measurable, where $\mathcal{S}$ is the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$.