Problem sheet 3

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1024020
until 22:00, May 6.

1. **[1+3 points]** Let $X = \mathbb{R}$. Consider the following semiring

$$H := \{(k, k+1] : \; k \in \mathbb{Z}\} \cup \{\emptyset\},$$

and define a measure μ on H as follows

$$\mu(\emptyset) := 0, \quad \mu((k, k+1]) := 1, \quad k \in \mathbb{Z}.$$

Let the measure $\bar{\mu}$ be the extension of μ to the ring $r(H)$ generated by H.

   (a) Compute $\bar{\mu}((0,1])$, $\bar{\mu}((1,2] \cup (5,6])$ and $\bar{\mu}((-1,3])$.

   (b) Construct the outer measure μ^* generated by $\bar{\mu}$ and compute $\mu^*\left(\left\{\frac{1}{2}\right\}\right)$, $\mu^*\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$ and $\mu^*(\mathbb{N})$.

2. **[2 points]** Let λ^* be an outer measure on 2^X. Show that a set $A \in 2^X$ is λ^*-measurable if and only if

$$\forall U \subset A \;\text{ and }\; \forall V \subset A^c \quad \lambda^*(U \cup V) = \lambda^*(U) + \lambda^*(V).$$

3. **[2 points]** Let μ^* be the outer measure generated by a measure μ defined on a ring R, and let $\mathcal{S}$ denotes the set of all μ^*-measurable sets. Show that $\sigma r(R) \subset \sigma(R) \subset \mathcal{S}$.

4. **[3+3 points]** Let $X = \mathbb{R}$ and λ be the Lebesgue measure. Denote by $\mathcal{S}$ the class of all Lebesgue measurable subsets of $\mathbb{R}$.

   (a) Let $A \in \mathcal{S}$, $\lambda(A) < +\infty$ and $f(x) := \lambda(A \cap (-\infty, x))$, $x \in \mathbb{R}$. Show that the function f is continuous on $\mathbb{R}$.

   (b) Let A be a bounded set and $\lambda(A) > 0$. Prove that there for every $\alpha \in (0, \lambda(A))$ there exists $B \subset A$, $B \in \mathcal{S}$ such that $\lambda(B) = \alpha$.

5. **[1+2+2 points]** Let $X = \mathbb{R}^2$ and λ be the Lebesgue measure. Denote by $\mathcal{S}$ the class of all Lebesgue measurable subsets of $\mathbb{R}^2$. Show that

   (a) a one-point set $\{(x,y)\}$ belongs to $\mathcal{S}$ and $\lambda(\{(x,y)\}) = 0$ for every $(x,y) \in \mathbb{R}^2$;

   (b) the interval $I = \{(x,y) : \; x \in [a,b], \; y = 1\}$ belongs to $\mathcal{S}$ and $\lambda(I) = 0$ for every $a < b$;

   (c) the line $L = \{(x,y) : \; x \in \mathbb{R}, \; y = 1\}$ belongs to $\mathcal{S}$ and $\lambda(L) = 0$;

   (d) **[3 bonus points]** the set $F = \{(x,y) : \; 0 \le x \le 1, \; 0 \le y \le f(x)\}$ belongs to $\mathcal{S}$ and $\lambda(F) = \int_0^1 f(x)dx$, where f is a nonnegative continuous function on $[0,1]$.

6. **[1+2 points]** Let $(X, \mathcal{F})$ and $(X', \mathcal{F}')$ be measurable spaces. Which functions $f : X \to X'$ are $(\mathcal{F}, \mathcal{F}')$-measurable if a) $\mathcal{F}' = \{\emptyset, X'\}$; b) $X = [0,1]$, $\mathcal{F} = \sigma(\{[0,1/2]\})$ and $X' = \mathbb{R}$, $\mathcal{F}' = \mathcal{B}(\mathbb{R})$.

7. **[3 bonus points]** Define the class of all μ^*-measurable sets, where μ^* is the outer measure from Exercise 1.

8. **[2 bonus points]** Fine an example of an outer measure λ^* on 2^X such that the class of all λ^*-measurable sets $\mathcal{S}$ equals $\{\emptyset, X\}$.