Problem sheet 2

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1012644
until 22:00, April 29.

1. **[1+1+1 points]** Let $X = \{0, 1, 2\}$ and $H = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. Is the function $\mu : H \to \mathbb{R}$ a measure on H, where μ is defined as
   (a) $\mu(\emptyset) := 0$, $\mu(\{0\}) = -1$, $\mu(\{1\}) = 2$ and $\mu(\{0, 1\}) = 1$;
   (b) $\mu(\emptyset) := 0$, $\mu(\{0\}) = 1$, $\mu(\{1\}) = 2$ and $\mu(\{0, 1\}) = 3$;
   (c) $\mu(\emptyset) := 0$, $\mu(\{0\}) = 1$, $\mu(\{1\}) = 2$ and $\mu(\{0, 1\}) = 1$?

2. **[3 points]** Show that a nonnegative, additive and continuous below[1] function μ on a ring H is a measure on H.

3. **[4 points]** Let μ be a measure on a σ-algebra $H \subset 2^X$ and $\mu(X) = 1$. For a family of sets $\{A_n : \ n \geq 1\} \subset H$ satisfying $\mu(A_n) = 1$, $n \geq 1$, show that $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 1$.

4. **[4 points]** Let μ be a measure on an algebra $H \subset 2^X$ and $\mu(X) = 1$. Let a family of sets $\{A_1, \dots, A_n\} \subset H$ satisfy the following property
$$\mu(A_1) + \dots + \mu(A_n) > n - 1.$$
   Show that $\mu\left(\bigcap_{k=1}^{n} A_k\right) > 0$.

5. **[4+4 points]** For a sequence $\{A_n : \ n \geq 1\}$ of subsets of X define
$$\varliminf_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k, \quad \varlimsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$[2]
   Let μ be a measure on a σ-algebra $H \subset 2^X$ and $\{A_n : \ n \geq 1\} \subset H$.
   (a) Prove that $\mu\left(\varliminf_{n\to\infty} A_n\right) \leq \varliminf_{n\to\infty} \mu(A_n)$.
   (b) Let additionally $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) < +\infty$. Show that $\mu\left(\varlimsup_{n\to\infty} A_n\right) \geq \varlimsup_{n\to\infty} \mu(A_n)$.

6. **[3 bonus points]** Show that a nonnegative, additive and σ-semiadditive function μ on a ring R is a measure on R.

7. **[4 bonus points]** Let μ be a measure on a σ-algebra $H \subset 2^X$ and $\mu(X) = 1$. Show that for every sequence $\{A_n : \ n \geq 1\} \subset H$ the equality
$$\sum_{n=1}^{\infty} \mu(A_n) < +\infty$$
   implies $\mu\left(\varlimsup_{n\to\infty} A_n\right) = 0$.

[1] A function μ defined on a ring H is called **continuous below**, if for every increasing family $\{A_n : \ n \geq 1\} \subset H$ one has $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$

[2] The set $\varliminf_{n\to\infty} A_n$ consists of all points which belongs to A_n for all $n \geq n_0$ starting from some nomber n_0. The set $\varlimsup_{n\to\infty} A_n$ consists of all point which belongs to an infinite number of A_n.