Problem sheet 10

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=1096518
until 22:00, June 24.

1. **[2 points]** If $x \perp y$ in an inner product space X, show that
$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

2. **[3 points]** Show that for a sequence $\{x_n\}_{n\geq 1}$ in an inner product space the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x\rangle \to \langle x, x\rangle$ imply the convergence $x_n \to x$.

3. **[4 bonus points]** Let H be a Hilbert space, $M \subset H$ be a convex subset, and $\{x_n\}_{n\geq 1}$ be a sequence in M such that $\|x_n\| \to d$, where $d = \inf_{x\in M} \|x\|$. Show that $\{x_n\}_{n\geq 1}$ converges in H.

   Hint: Use the parallelogram equality

4. **[4 points]** Let $\{e_1, \dots, e_n\}$ be an orthonormal set in an inner product space X, where n is fixed. Let $x \in X$ be any fixed element and $y = \beta_1 e_1 + \dots + \beta_n e_n$. Then $\|x - y\|$ depends on $\beta_1, \dots, \beta_n$. Show by direct calculation that $\|x - y\|$ is minumum if and only if $\beta_k = \langle x, e_k\rangle$, $k = 1, \dots, n$.

5. **[4 points]** Let $x_1(t) = t^2$, $x_2(t) = t$, $x_3(t) = 1$. Orthonormalize x_1, x_2, x_3 in this order in the space $L^2[-1, 1]$.

6. **[5 points]** Let M be a total set in an inner product space X. If $\langle v, x\rangle = \langle w, x\rangle$ for all $x \in M$, show that $v = w$.

7. **[5 points]** Let $\{e_k,\ k \geq 1\}$ be an orthonormal sequence in a Hilbert space H. Show that if
$$x = \sum_{k=1}^{\infty} \alpha_k e_k, \quad y = \sum_{k=1}^{\infty} \beta_k e_k,$$
then
$$\langle x, y\rangle = \sum_{k=1}^{\infty} \alpha_k \bar{\beta}_k$$
and the series converges absolutely.

8. **[4 bonus points]** Let $\{e_k,\ k \geq 1\}$ be an orthonormal sequence in a Hilbert space H, and let $M = \text{span}\,\{e_k,\ k \geq 1\}$. Show that for any $x \in H$ we have that $x \in \bar{M}$ if and only if x can be represented as
$$x = \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k.$$