Problem sheet 1

Solutions has to be uploaded into Moodle:
https://moodle2.uni-leipzig.de/mod/assign/view.php?id=995541
until 22:00, April 22.

1. **[2 points]** Prove that a nonempty class of sets $H \subset 2^X$ is a ring if and only if H is a semiring and $A \cup B \in H$ for every $A, B \in H$.

2. **[2 points]** Let H be a ring. Show that H is a semiring.

3. **[3 points]** Let $X = \mathbb{R}$ and the class $H \subset 2^X$ consists of all subsets with finite number of elements. Show that H is a ring. Is H a σ-ring? Justify your answer.

4. **[4 points]** Let X be an uncountable set, and let H be the class of subsets of X, which either are at most countable or have at most countable complements. Prove or disprove that H is a σ-algebra.

   Hint: Use here the result of Exercise 11.

5. **[1+1 point]** Let $H_1, H_2, H \subset 2^X$.

   (a) Show that the inclusion $H_1 \subset H_2$ implies $\sigma(H_1) \subseteq \sigma(H_2)$.

   (b) Let $H \subset 2^X$. Show that $\sigma(H) = \sigma(\sigma(H))$.

6. **[5 points]** Let $B \subset X$ be fixed and $H \subset 2^X$. Show that $\sigma r(H \cap B) = \sigma r(H) \cap B$. Here $H \cap B = \{A \cap B : A \in H\}$ and $\sigma r(H) \cap B = \{A \cap B : A \in \sigma r(H)\}$.

7. **[2 points]** Let $X = [0, 2]$ and $H = \{\{0\}, [0, 1)\}$. Construct $r(H)$ and $a(H)$.

8. **[2 points]** Let $H = \{(a, b] : -\infty < a < b < +\infty\} \cup \{\emptyset\}$. Show that $\sigma(H) = \mathcal{B}(\mathbb{R})$.

9. **[4 bonus points]** Show that there exists a class H consisting of countable number of sets from $\mathbb{R}^2$ such that $\mathcal{B}(\mathbb{R}^2) = \sigma(H)$.

10. **[5 bonus points]** Prove the Theorem 2.18 from Lecture note 2. Namely, let the class H consists of all open sets from $\mathbb{R}$. Show that $\mathcal{B}(\mathbb{R}) = \sigma(H)$.

11. **[2 bonus points]** Let for every $n \geq 1$ a set A_n contains countable number of elements. Show that $\bigcup_{n=1}^{\infty} A_n$ also contains a countable number of elements.