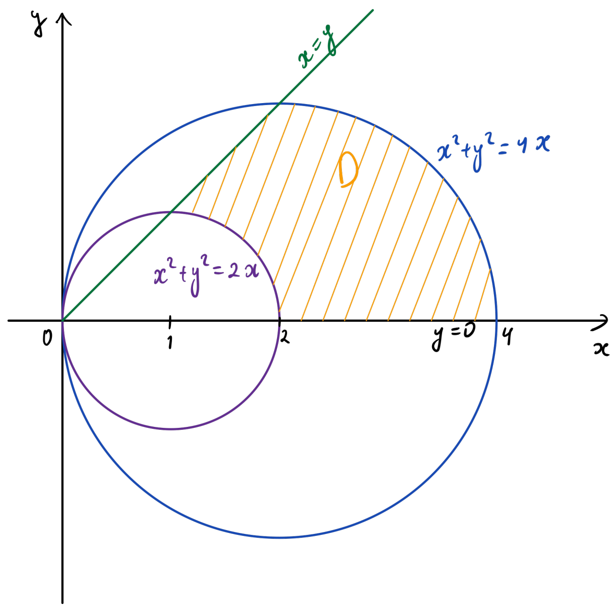
Exam Solutions

Each exercise is graded between 0 and 5 points.

1. Passing to polar coordinates, find the area bounded by the lines

$$x^2 + y^2 = 2x, \quad x^2 + y^2 = 4x, \quad y = x, \quad y = 0.$$

Solution. We first remark that $x^2 + y^2 = 2x$ is a circle with centers $(1, 0)$ and radius 1, since the equation can be rewritten as $(x - 1)^2 + y^2 = 1$. Similarly, $x^2 + y^2 = 4x$ is a circle with center $(2, 0)$ and radius 2.



Passing to the polar coordinate

$$x = \rho \cos\varphi, \quad y = \rho \sin\varphi, \quad 0 \le \varphi \le 2\pi, \quad \rho \ge 0,$$

we can write the equation for the circle $x^2 + y^2 = 2x$ in the new coordinates as follows:

$$\begin{aligned} \rho^2 \cos\varphi + \rho^2 \sin\varphi &= 2\rho\cos\varphi, \\ \rho^2 &= 2\rho\cos\varphi, \\ \rho &= 2\cos\varphi, \quad 0 \le \varphi \le 2\pi. \end{aligned}$$

Similarly, for the second circle we have

$$\rho = 4\cos\varphi, \quad 0 \le \varphi \le 2\pi.$$

Hence, the area $S(D)$ of the region D between the given lines equals

$$\begin{aligned}\text{Area}(D) &= \iint_D dxdy = \int_0^{\frac{\pi}{4}} d\varphi \int_{2\cos\varphi}^{4\cos\varphi} \rho\, d\rho = \int_0^{\frac{\pi}{4}} \left(\frac{\rho^2}{2}\Big|_{\rho=2\cos\varphi}^{\rho=4\cos\varphi} \right) d\varphi \\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(16\cos^2\varphi - 4\cos^2\varphi\right) d\varphi = 6\int_0^{\frac{\pi}{4}} \frac{1+\cos 2\varphi}{2} d\varphi \\ &= 3\int_0^{\frac{\pi}{4}} (1+\cos 2\varphi)\, d\varphi = 3\left(\varphi + \frac{\sin 2\varphi}{2}\right)\Big|_0^{\frac{\pi}{4}} = 3\left(\frac{\pi}{4}+\frac{1}{2}\right) = \frac{3}{4}(\pi+2).\end{aligned}$$

2. Evaluate the line integral

$$\int_\gamma xyds,$$

where γ is a quarter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lying in the first quadrant.

Solution. First we write a parametrisation for the quarter of the ellipse

$$\begin{aligned}x &= a\cos t,\\ y &= b\sin t, \quad 0 \le t \le \frac{\pi}{2}.\end{aligned}$$

Then

$$\begin{aligned}\int_\gamma xyds &= \int_0^{\frac{\pi}{2}} ab\cos t\sin t\sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt \\ &= ab\int_0^{\frac{\pi}{2}} \sin t\sqrt{a^2\sin^2 t + b^2(1-\sin^2 t)}\, d\sin t \overset{\sin t = u}{=} ab\int_0^1 u\sqrt{a^2u^2 + b^2(1-u^2)}\, du \\ &= \frac{ab}{2}\int_0^1 \sqrt{b^2 + (a^2-b^2)u^2}\, du^2 \overset{u^2=v}{=} \frac{ab}{2}\int_0^1 \sqrt{b^2+(a^2-b^2)v}\, dv = \\ &= \frac{ab}{2(a^2-b^2)}\int_0^1 (b^2+(a^2-b^2)v)^{\frac{1}{2}}\, d(b^2+(a^2-b^2)v) \\ &= \frac{ab}{2(a^2-b^2)} \frac{2(b^2+(a^2-b^2)v)^{\frac{3}{2}}}{3}\Big|_0^1 = \frac{ab(a^3-b^3)}{3(a^2-b^2)},\end{aligned}$$

if $a \ne b$. If $a = b$, than

$$\int_\gamma xyds = \dots = \frac{ab}{2}\int_0^1 \sqrt{b^2+(a^2-b^2)v}dv = \frac{a^2}{2}\int_0^1 \sqrt{a^2}dv = \frac{a^3}{2}$$

3. Evaluate the following surface integral

$$\iint_S x^3dydz + y^3dzdx + z^3dxdy,$$

where S is the sphere $x^2+y^2+z^2 = a^2$ $(a > 0)$ oriented inward.

Solution. Let V denote the ball with center at $(0,0,0)$ and radius a. Using the Gauss-Ostrogradskii formula, we obtain

$$\iint_S x^3dydz+y^3dzdx+z^3dxdy = -\iiint_V \text{div}(x^3,y^3,z^3)dxdydz = -3\iiint_V \left(x^2+y^2+z^2\right) dxdydz.$$

We have "–" before the domain integral because of the orientation is inward.

Passing to the spherical coordinates

$$x = \rho \cos\varphi \cos\psi, \quad y = \rho \sin\varphi \cos\psi, \quad z = \rho \sin\psi,$$
$$J = \rho^2 \cos\psi,$$

we have

$$-3 \iiint_V \left(x^2 + y^2 + z^2\right) dxdydz = -3 \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^a \rho^2 \rho^2 \cos\psi d\rho$$
$$= -3 \cdot 2\pi \cdot \sin\psi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \frac{\rho^5}{5}\Big|_0^a = -\frac{12\pi a^5}{5}.$$

4. Applying Stokes' formula, find the integral

$$\int_\gamma (y - z)dx + (z - x)dy + (x - y)dz,$$

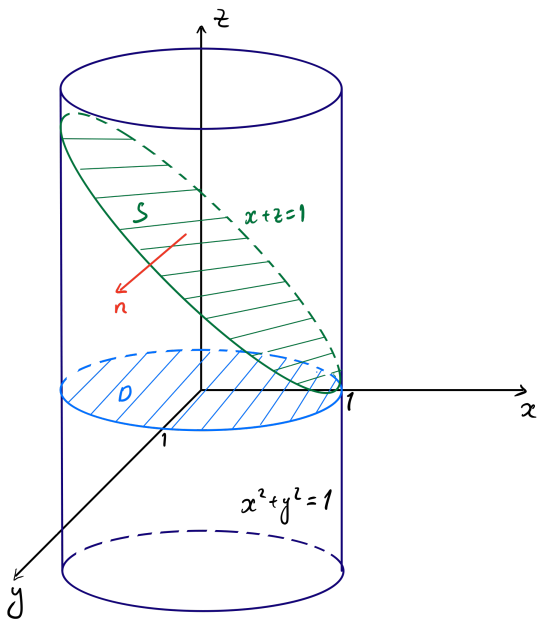
where γ is the ellipse $x^2 + y^2 = 1$, $x + z = 1$, oriented counter clockwise viewed from the point $(0, 0, 0)$.

Solution. By Stokes' formula

$$\int_\gamma F \cdot ds = \iint_S \operatorname{curl} F \cdot dS = \iint_S (\operatorname{curl} F \cdot n)\, dS,$$

where $F = (y - z, z - x, x - y)$, $n = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$ and

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x & x - y \end{vmatrix} = -(2, 2, 2).$$



So, using the parametrization $z = 1 - x$, $(x, y) \in D = \{(x, y) : \ x^2 + y^2 \le 1\}$, for S, we obtain

$$\int_\gamma (y-z)dx + (z-x)dy + (x-y)dz = -\iint_S (2,2,2)\cdot\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) dS$$
$$= 2\sqrt{2}\iint_S dS = 2\sqrt{2}\iint_D \sqrt{1 + (z'_x)^2 + (z'_y)^2}dxdy$$
$$= 2\sqrt{2}\iint_D \sqrt{1 + (-1)^2 + 0} = 2\sqrt{2}^2 \iint_D dxdy = 4\cdot\text{Area}(D) = 4\pi.$$

5. Let $f(z) = \bar{z}^2$, $z \in \mathbb{C}$. Is f differentiable at 0? Is f holomorphic at 0? Justify you answer.

   Solution. The function $f(z) = \bar{z}^2$ is differentiable at 0 because

$$\lim_{z\to 0}\frac{f(z) - f(0)}{z} = \lim_{z\to 0}\frac{\bar{z}^2}{z} = \lim_{r\to 0}\frac{r^2(\cos\varphi - i\sin\varphi)}{r(\cos\varphi + i\sin\varphi)} = \lim_{r\to 0}\frac{r(\cos\varphi - i\sin\varphi)^2}{\cos^2\varphi + \sin^2\varphi} = 0.$$

   But the function $f(z) = f(x+iy) = (x-iy)^2 = x^2 - y^2 - 2ixy$ is not holomorphic at 0. Indeed, according to the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial(x^2 - y^2)}{\partial x} = 2x \ne \frac{\partial v}{\partial y}(x,y) = \frac{\partial(-2xy)}{\partial y} = -2x$$

   for all $x \ne 0$. So, the function f is not differentiable in a neighbourhood of 0.

6. Find the Laurent series for the function $f(z) = \frac{1}{(z-2)(z+1)^2}$ in $0 < |z+1| < 3$.

   Solution. We will use the expansion $\frac{1}{1-z} = \sum_{n=0}^\infty z^n$, $|z| < 1$.

$$f(z) = \frac{1}{(z-2)(z+1)^2} = \frac{1}{(z+1)^2}\cdot\frac{1}{-3 + (z+1)} = -\frac{1}{3(z+1)^2}\cdot\frac{1}{1 - \frac{z+1}{3}}$$
$$= -\frac{1}{3(z+1)^2}\sum_{n=0}^\infty \frac{(z+1)^n}{3^n} = -\sum_{n=0}^\infty\frac{(z+1)^{n-2}}{3^{n+1}}, \quad 0 < |z+1| < 3.$$

7. Compute the integral

$$\int_{|z-2|=2} z^2\left(\cos^2\frac{1}{z-1} + \frac{1}{(4-z^2)^2}\right)dz.$$

   Solution. Consider separately $\int_{|z-2|=2} z^2\cos^2\frac{1}{z-1}dz$ and $\int_{|z-2|=2}\frac{z^2}{(4-z^2)^2}dz$. By the residue theorem

$$\int_{|z-2|=2} z^2\cos^2\frac{1}{z-1}dz = 2\pi i\,\text{res}_1\, z^2\cos^2\frac{1}{z-1}.$$

   The point $z = 1$ is an essential isolated singularity for the function $z^2\cos^2\frac{1}{z-1}$. We find its residue at $z = 1$ by computing c_{-1} in the Laurent series of $z^2\cos^2\frac{1}{z-1}$ around the point $z = 1$. So,

$$z^2\cos^2\frac{1}{z-1} = ((z-1)+1)^2\left(1 - \frac{1}{2!(z-1)^2} + \frac{1}{4!(z-1)^4} + \dots\right)^2$$
$$= \left((z-1)^2 + 2(z-1) + 1\right)\left(1 - \frac{2}{2!(z-1)^2} + \left(\frac{2}{4!} + \frac{1}{(2!)^2}\right)\frac{1}{(z-1)^4} + \dots\right)$$

Hence $c_{-1} = 2 \cdot \frac{-2}{2!} = -2$.

Similarly,

$$\int_{|z-2|=2} \frac{z^2}{(4-z^2)^2} dz = 2\pi i \operatorname{res}_2 \frac{z^2}{(4-z^2)^2}.$$

Here, the point $z = 2$ is a pole of order 2 for the function $\frac{z^2}{(4-z^2)^2} = \frac{z^2}{(2-z)^2(2+z)^2}$. Hence

$$\begin{aligned}\operatorname{res}_2 \frac{z^2}{(4-z^2)^2} &= \lim_{z\to 2} \frac{d}{dz}\left((z-2)^2 \frac{z^2}{(2-z)^2(2+z)^2}\right) = \lim_{z\to 2} \frac{d}{dz} \frac{z^2}{(2+z)^2} \\ &= \lim_{z\to 2} \frac{2z(2+z)^2 - 2z^2(2+z)}{(2+z)^4} = \frac{1}{8}.\end{aligned}$$

Thus,

$$\int_{|z-2|=2} z^2 \left(\cos^2 \frac{1}{z-1} + \frac{1}{(4-z^2)^2}\right) dz = 2\pi i \left(-2 + \frac{1}{8}\right) = -\frac{15\pi i}{4}.$$

8. Solve the heat equation

$$\begin{aligned}&u_t(t,x) = \frac{1}{4} u_{xx}(t,x), \quad x \in \mathbb{R}, \quad t > 0, \\ &u(0,x) = x^2, \quad x \in \mathbb{R}.\end{aligned}$$

Solution. The solution to the heat equation is given by the formula

$$u(t,x) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{t}} y^2 dy.$$

We change the variables $\frac{y-x}{\sqrt{t}} = \alpha$. Then $y = \sqrt{t}\alpha + x$ and $dy = \sqrt{t} d\alpha$. So,

$$\begin{aligned}u(t,x) &= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\alpha^2} (\sqrt{t}\alpha + x)^2 \sqrt{t} d\alpha = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} \left(t\alpha^2 + 2\sqrt{t}x\alpha + x^2\right) d\alpha \\ &= \frac{t}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} \alpha^2 d\alpha + \frac{2\sqrt{t}x}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} \alpha d\alpha + \frac{x^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha \\ &= -\frac{t}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \alpha d e^{-\alpha^2} + 0 + x^2 = -\frac{t}{2\sqrt{\pi}} \left(\alpha e^{-\alpha^2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha\right) + x^2 = \frac{t}{2} + x^2.\end{aligned}$$

We have used here $\int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha = \sqrt{\pi}$, and $\int_{-\infty}^{\infty} e^{-\alpha^2} \alpha d\alpha = 0$ because the function $e^{-\alpha^2}\alpha$ is odd.