25. Heat equation

1. Fourier transform on $\mathbb{R}$

Def 25.1 The Fourier transform of a continuous, absolutely integrable function $f:\mathbb{R}^d \to \mathbb{C}$ is defined by

$$\boxed{\hat{f}(\sigma) = \mathcal{F}[f](\sigma) = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{-i\sigma\cdot x} f(x)\,dx,}$$

where $\sigma\cdot x = \sigma_1 x_1 + \ldots + \sigma_d x_d$

Th 25.1 Let a function f and $\hat{f}$ be absolutely integrable. Then

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{i\sigma\cdot x} \hat{f}(\sigma)\,d\sigma.$$

Remark 25.1 From Theorem 25.1 it follows that

$$\mathcal{F}^{-1}[\mathcal{F}[f]] = f \quad \text{and}$$

$$\mathcal{F}[\mathcal{F}^{-1}[g]] = g.$$

Next, we assume that function f is differentiable, then

$$\frac{\partial}{\partial x_1} f(x) = \frac{\partial}{\partial x_1} \mathcal{F}^{-1}[\hat{f}](x) =$$

$$= \frac{\partial}{\partial x_1} \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{i\sigma\cdot x} \hat{f}(\sigma)\,d\sigma =$$

$$= \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} i\sigma_1 e^{i\sigma\cdot x} \hat{f}(\sigma)\, d\sigma =$$

$$= \mathcal{F}^{-1}[\, i\sigma_1 \hat{f}(\sigma)\,]$$

Hence,

$$\mathcal{F}\left[\frac{\partial}{\partial x_1} f\right](\sigma) = i\sigma_1 \hat{f}(\sigma)$$

A similar computations gives

$$\boxed{\begin{aligned} \mathcal{F}[D^\alpha f] &= (i\sigma)^\alpha \mathcal{F}[f] \\ D^\alpha \mathcal{F}[f] &= \mathcal{F}[(-ix)^\alpha f] \qquad (25.1) \end{aligned}}$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$, $|\alpha| = \alpha_1 + \dots + \alpha_d$,

$\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$

and $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$.

For two functions φ, ψ we define the **convolution** as

$$f * g\,(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

Exercise 25.1 Show that

$$\mathcal{F}[f * g] = \sqrt{(2\pi)^d}\, \mathcal{F}[f]\, \mathcal{F}[g]$$

and

$$\mathcal{F}[fg] = \frac{1}{(2\pi)^d} \mathcal{F}[f] * \mathcal{F}[g].$$

2. Heat equation on $\mathbb{R}$

In this sectin, we are going to solve the following equation

$$\begin{cases} u_t = a^2 u_{xx} + f(t,x) & t>0,\ x \in \mathbb{R}, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}. \end{cases} \tag{25.2}$$

In order to find a solution to (25.2) we first do formal computations. We take the Fourier transfor of the left and right hand of the equation. We obtain

$$F[u_t] = F[a^2 u_{xx} + f(t,x)]$$

By (25.1), we obtain

$$\frac{d}{dt} F[u(t,\cdot)](\sigma) = a^2 (i\sigma)^2 F[u(t,\cdot)](\sigma) + \hat{f}(t,\sigma)$$

Denote

$$v(t,\sigma) := F[u(t,\cdot)](\sigma).$$

Then we have obtained the equation for v:

$$\frac{d}{dt} v(t,\sigma) = -a^2\sigma^2 v(t,\sigma) + \hat{f}(t,\sigma). \quad (25.3)$$

Here $\sigma \in R$ is a parameter. We note that we have obtain a linear ordinary differential equation.

Next, we take the Fourier transform of the left and right part of the initial condition

$$u(0,x) = \varphi(x).$$

We obtain

$$v(0,\sigma) = \hat{\varphi}(\sigma) \quad (25.4)$$

Solving (25.4), we have

$$(25.5) \quad v(t,\sigma) = e^{-a^2\sigma^2 t}\hat{\varphi}(\sigma) + \int_0^t e^{-a^2\sigma^2(t-s)}\hat{f}(s,\sigma)\,ds$$

Since $v(t,\sigma) = F[u(t,\cdot)](\sigma)$, we can take the inverse Fourier transform of the left and right hand sides of (25.5). Thus,

$$u(t,\cdot) = \mathcal{F}^{-1}[e^{-a^2\sigma^2 t}\hat{\varphi}] + \int_0^t \mathcal{F}^{-1}[e^{-a^2\sigma^2(t-s)}\hat{f}(s,\cdot)]\,ds$$

$$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[e^{-a^2\sigma^2 t}] * \mathcal{F}^{-1}[\hat{\varphi}] +$$

$$+\frac{1}{\sqrt{2\pi}}\int_0^t \mathcal{F}^{-1}[e^{-a^2\sigma^2(t-s)}]*\mathcal{F}^{-1}[\hat{f}(s,\cdot)]\,ds =$$

$$=\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[e^{-a^2\sigma^2 t}]*\varphi+\frac{1}{\sqrt{2\pi}}\int_0^t \mathcal{F}^{-1}[e^{-a^2\sigma^2(t-s)}]*f(s,\cdot)\,ds$$

Exercise 25.2 Show that

$$\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[e^{-a^2\sigma^2 t}]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\sigma x}e^{-a^2\sigma^2 t}\,d\sigma=$$

$$=\boxed{\frac{1}{\sqrt{4\pi a^2 t}}e^{-\frac{x^2}{4a^2t}}=:G(t,x)}$$

Hence, we have obtained the formula:

$$u(t,x)=\int_{-\infty}^{+\infty}G(t,x-y)\varphi(y)\,dy+$$

$$+\int_0^t\int_{-\infty}^{+\infty}G(t-s,x-y)f(s,y)\,dy\,ds$$

– solution to the equation (25.2)

3. Heat equation on an interval $[0,l]$

In this section, we consider the following equation

$$u_t(t,x) = a^2 u_{xx}(t,x) + f(t,x), \quad t>0,\ x\in(0,l),$$

$$u(0,x) = \varphi(x), \quad x \in (0,l)$$

We also need the boundary conditions

$$\begin{cases} u(t,0) = \nu_1(t) \\ u(t,l) = \nu_2(t) \end{cases}$$

– Dirichlet boundary conditions

or

$$\begin{cases} u_x(t,0) = \mu_1(t) \\ u_x(t,l) = \mu_2(t) \end{cases}$$

– Neumann b.c.

(There could be mixed boundary conditions, like $u(t,0) = \nu_1(t)$, $u_x(t,l) = \mu_2(t)$)

We demonstrate the method of solving on concrete example.

Ex 25.1 We consider the equation

$$(25.5) \qquad u_t = a^2 u_{xx} + \overbrace{\cos\frac{3\pi}{2l}x}^{=:f}, \quad t>0,\ x\in(0,l)$$

$$(25.6) \qquad \begin{cases} u_x(t,0) = 0, \\ u(t,l) = 0, \end{cases} \quad t \geq 0$$

– boundary cond.

$$(25.7) \qquad u(0,x) = A(l-x), \quad x \in [0,l]$$

← initial condition

1) We first find a solution to (25.5) in the form

$$u(x,t) = X(x)T(t) \longrightarrow (25.5)$$ (with $f=0$)

We obtain

$$T'(t)X(x) = a^2 T(t) X''(x) \quad /: a^2TX$$

$$\frac{T'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$ (left hand side depends on t, right hand side depends on x, so both of them must be a constant.)

Hence, we obtain the equation for X:

$$X''(x) + \lambda X(x) = 0 \qquad (25.8)$$

Next, plug in u into (25.6)
(they must be zero boundary conditions)

So

$$T(t)X'(0) = 0$$
$$T(t)X(\ell) = 0$$

Hence $X'(0) = 0, \quad X(\ell) = 0 \qquad (25.9)$

– boundary conditions for (25.8).

2) Find non zero solutions to (25.8)(25.9).

The problem (25.8), (25.9) is called Sturm - Liouville problem.

(25.8) is a linear second order equation.
To find its solution, we need to find roots of the characteristic polynomial

$$\mu^2 + \lambda = 0.$$

a) $\lambda < 0 \Rightarrow \mu = \pm \sqrt{-\lambda}$

$$X(x) = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x}$$

From (25.9)

$$\begin{cases} X(0) = c_1 + c_2 = 0 \\ X(l) = c_1 e^{\sqrt{-\lambda} l} + c_2 e^{-\sqrt{-\lambda} l} = 0. \end{cases}$$

This sistem has only solution $c_1 = c_2 = 0$.

b) $\lambda = 0 \Rightarrow \mu = 0$

$$X(x) = c_1 x + c_2.$$

Similarly, from (25.9) $c_1 = c_2 = 0$.

c) $\lambda > 0$

$$\mu = \pm i\sqrt{\lambda}.$$

So,

$$X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$$

From (25.8):

$$X'(x) = -C_2\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$X'(0) = C_2\sqrt{\lambda} = 0 \Rightarrow C_2 = 0.$$

Next, $X(l) = C_1 \cos\sqrt{\lambda}x = 0 \Rightarrow$

$$\sqrt{\lambda}l = \frac{\pi}{2} + \pi n, \quad n = 0, 1, 2, \dots$$

$$\sqrt{\lambda_n} = \frac{\pi(1+2n)}{2l}, \quad n = 0, 1, 2, \dots$$

We obtained nonzero solutions to (25.8)(25.9):

$$X_n(x) = \cos\frac{\pi(1+2n)}{2l}x.$$