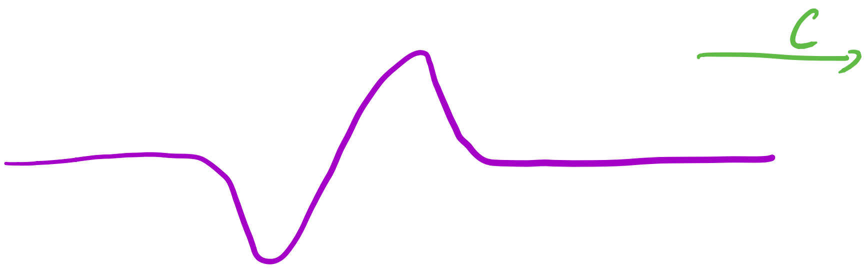
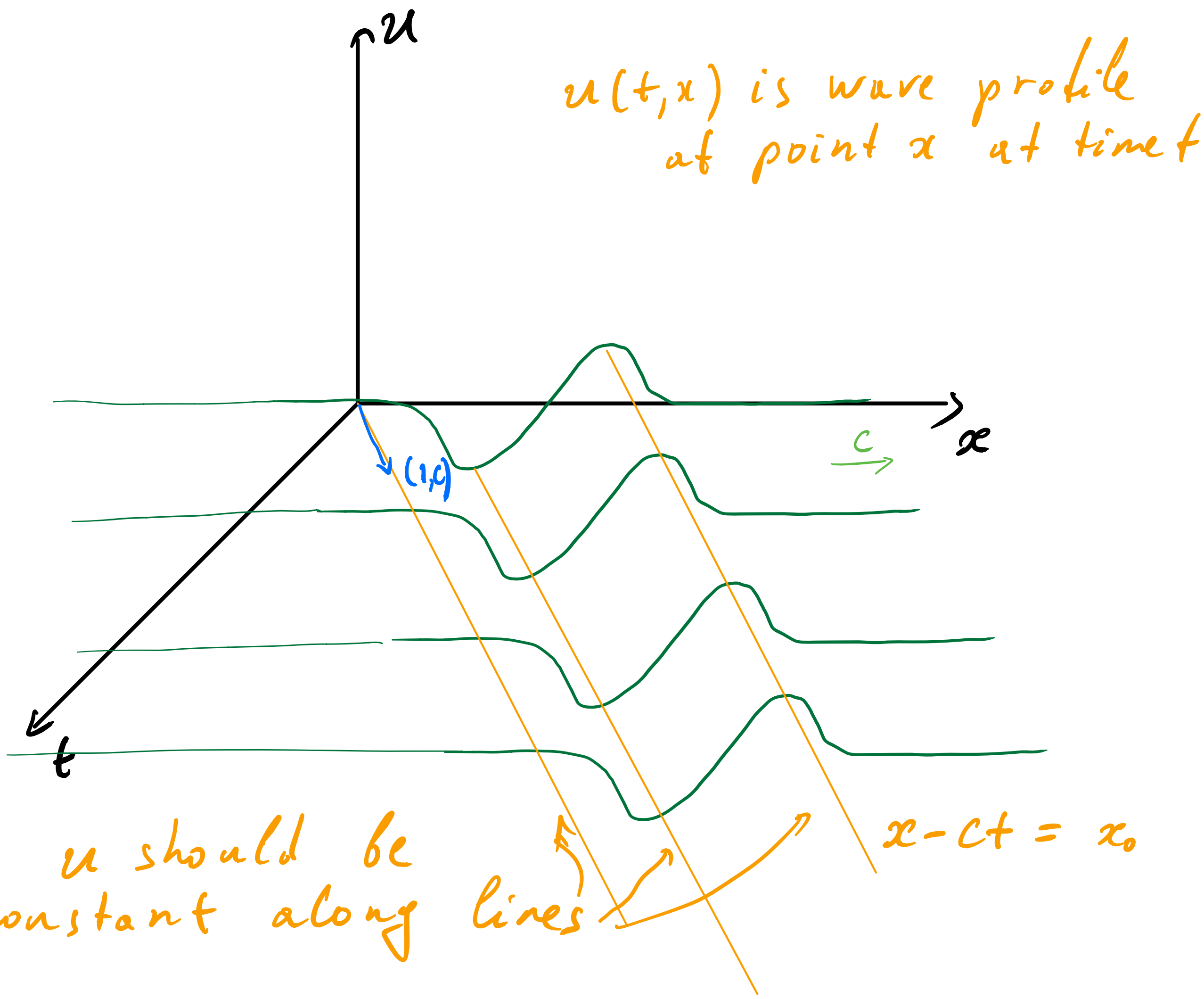
24. Introduction to PDE. Transport equation

1. Transport equation

Let us assume that we have a wave



which moving with a constant speed c
How we can describe the motion of traviling wave. Let us consider a picture



The lines $x - ct = x_0$, where u is constant on, are cald characteristic lines.
This implies that directional derivative of u in direction of $x - ct = x_0$ equal 0.
So, for $\ell = (1, c)$

$$\frac{\partial u}{\partial \ell} = (1, c) \cdot \nabla u = (1, c)(u_t, u_x) =$$

$$= u_t + c\, u_x = 0,$$

where $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$.

Hence, we obtained the equation

$$(24.1) \quad \boxed{\begin{array}{ll} u_t + c\, u_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = f(x), & x \in \mathbb{R}. \end{array}}$$

– equation
– initial condition

The obtained equation is called a transport equation with constant coefficients.

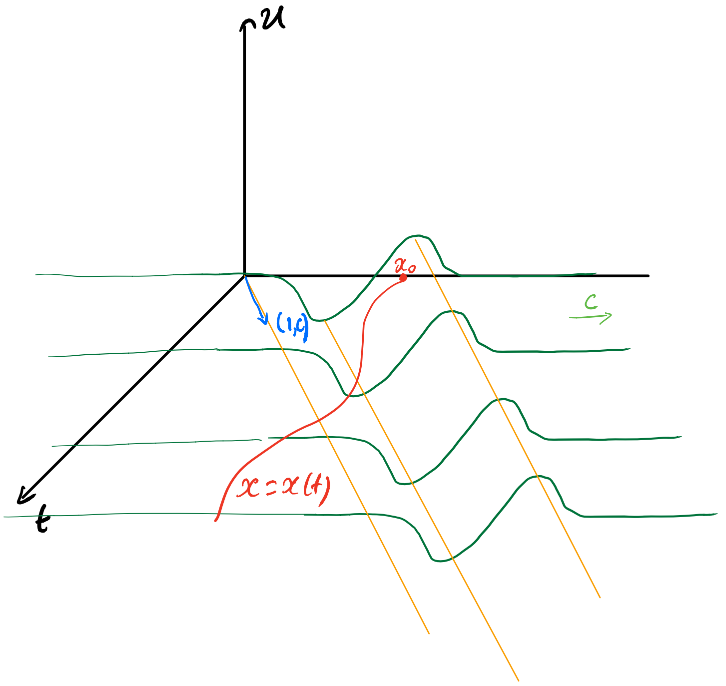
Next, we are going to find solution to (24.1), that is, to find a function

$$u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$$

which is differentiable in t and x and satisfies (24.1).

Method of characteristics

We assume that $x = x(t)$ (we can interpret it as a coordinate of moving observer)



Then

$u(t, x(t))$ – the point which observer sees at time t.

Let us compute the derivative of $u(t, x(t))$

$$\frac{d}{dt} u(t, x(t)) = u_t + \frac{dx}{dt} u_x$$

Then u satisfies the equation (24.1) if

observer is moving with speed c

$$\frac{dx}{dt} = c \, , \quad \frac{d}{dt} u(t, x(t)) = 0$$

u is not changing along $x = x(t)$

or $\frac{d}{dt} u = 0$ along $\frac{dx}{dt} = c$

We solve obtained equations

$$x = ct + x_0 \quad \text{or} \quad x_0 = x - ct$$

Hence

$$u(t, x(t)) = u(0, x(0)) = f(x_0)$$

Hence, $$\boxed{u(t,x) = f(x-ct)}$$

is a solution to (24.1).

Now, we show that the equation has no other solutions. Let u satisfies (24.1). We consider a new function

$$v(t,x) = u(t, x+ct)$$

Then

$$v_t(t,x) = u_t(t, x+ct) + c u_x(t, x+ct) = 0$$

Hence

$$V(t,x) = F(x) \text{ — some function}$$

But

$$v(0,x) = u(0, x+c\cdot 0) = u(0,x) = f(x)$$

So, $F(x) = f(x)$. Consequently

$$v(t,x) = f(x)$$

and $u(t,x) = v(t, x-ct) = f(x-ct)$.

Ex 24.1 We solve the equation

$$\begin{cases} u_t + 2u_x = 0 \\ u(0,x) = \cos x \end{cases}$$

We use method of characteristics:

$$\frac{dx}{dt} = 2$$

$$x = 2t + x_0 \quad \text{or} \quad x_0 = x - 2t.$$

Consequently,

$$u(x,t) = \cos(x-2t)$$

Remark 24.1 The same method works in the case of the equation

$$a(t,x)\, u_t + b(t,x)\, u_x = 0.$$

Or deviding by $a(t,x)$ we can rewrite the equation as

$$u_t + c(t,x)\, u_x = 0.$$

Ex 24.2 We solve the equatoon

$$\begin{cases} u_t - (x+1)\, u_x = 0 \\ u(0,t) = f(x) \end{cases}$$

We write

$$\frac{dx}{dt} = -(x+1)$$

$$\frac{dx}{x+1} = -\, dt$$

$$\int \frac{dx}{x+1} = -t$$

$$\ln|x+1| = -t$$

$$x+1 = c\,e^{-t}$$

$$x = c\,e^{-t} + 1$$

$$x(0) = c + 1 = x_0 \Rightarrow c = x_0 - 1$$

Consequently
$$x = (x_0 - 1)e^{-t} + 1$$

Find x_0:
$$x_0 = (x-1)e^{t} + 1$$

Hence,
$$u(t,x) = f\big((x-1)e^{t}+1\big).$$

2. Partial differential equations. Fundamental examples.

Def 24.1 A *partial differential equation* (PDE) of a single unknown u is an equation involving u and its partial derivatives. All such equations can be written as

$F(u, u_{x_1}, \dots, u_{x_n}, u_{x_1 x_1}, \dots, u_{x_{i_1} \dots x_{i_N}}, x_1, \dots, x_n) = 0$
for some function F.

Here N is called the order of the PDE.
(N is the maximum number of derivatives appering in the equations.

Ex 24.3 (Heat equation (from class od parabolic equation))

We will talk about classes of second order equations later.

$$\boxed{u_t = a^2 u_{xx}}$$

Here t represents time and x is a spetial coordinate and
$u(t, x)$ is a temperature at point x at time t.

The equation descrube the conductance of temperature at a metal wire.

Ex 24.4 (wave equation (from class of hyperbolic equations))

$$\boxed{u_{tt} = a^2 u_{xx}}$$

Again t describe the time and x is a spetial variable.

$u(t,x)$ can be interpeted as the high of profile of water wave or as a position of vibration string at point x at time t.

Ex 24.5 (Laplace equation (from class of elliptic equations))

$$\boxed{u_{xx} + u_{yy} = 0}$$

Here x, y are spetial varoables.

The equation can describe mechanical equilibrium or temperature equilibrium

3. Fourier transform on $\mathbb{R}^d$.

Def 24.2 The Fourier transform of a continuous, absolutely integrable function $f: \mathbb{R}^d \to \mathbb{C}$ is defined by

$$\boxed{\hat{f}(\sigma) = \mathcal{F}[f](\sigma) = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{-i\sigma \cdot x} f(x)\, dx,}$$

where $\sigma \cdot x = \sigma_1 x_1 + \ldots + \sigma_d x_d$