23. Application of res. to computation of integrals

1. Computations of residues

1) a is removable singularity of f

$$\boxed{\operatorname{res}_a f = 0}$$

2) a is a pole of order 1

$$\boxed{\operatorname{res}_a f = \lim_{z\to a} (z-a) f(z)}$$

Indeed,

$$f(z) = \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n.$$

So,

$$\lim_{z\to a} (z-a) f(z) = \lim_{z\to 0}\left(c_{-1} + c_0(z-a) + \dots\right) =$$

$$= c_{-1} = \operatorname{res}_a f.$$

3) $f(z) = \dfrac{\varphi(z)}{\psi(z)}$, where φ, ψ are holomorphic,

$\psi(a) = 0$, $\psi'(a) \neq 0$, $\varphi(a) \neq 0$

$$\boxed{\operatorname{res}_a \frac{\varphi}{\psi} = \frac{\varphi(a)}{\psi'(a)}}$$

Indeed, by 2)

$$\operatorname{res}_a \frac{\varphi}{\psi} = \lim_{z\to a} \frac{(z-a)\varphi(z)}{\psi(z) - \underbrace{\psi(a)}_{=0}} = \frac{\varphi(a)}{\psi'(a)}.$$

4) a is a pole of f of order n

$$\boxed{\operatorname{res}_a f = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}}\Big((z-a)^n f(z)\Big)}$$

In order to prove the equality, we will write

$$f(z) = \frac{C_{-n}}{(z-a)^n} + \frac{C_{-n+1}}{(z-a)^{n-1}} + \dots +$$

$$+ \frac{C_{-1}}{z-a} + C_0 + \dots$$

$$(z-a)^n f(z) = C_{-n} + C_{-n+1}(z-a) + \dots +$$

$$+ C_{-1}(z-a)^{n-1} + C_0 (z-a)^n + \dots$$

Hence, $$\frac{d^{n-1}}{dz^{n-1}}\Big((z-a)^n f(z)\Big) = (n-1)!\, C_{-1} + \tilde{C}_0 (z-a) + \dots$$

So,

$$\lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}}\Big((z-a)^n f(z)\Big) = (n-1)!\, C_{-1}.$$

Exercise 23.1 Check that z is a pole of f of order N iff there exists a holomorphic function φ in a neighbourhood of a such that

$$f(z) = \frac{\varphi(z)}{(z-a)^N}$$

and $\varphi(a) \neq 0$.

Ex 23.1 **a)** $f(z) = \dfrac{z}{(z-1)(z-2)^2}$

$z=1$, $z=2$ are isolated singular points which are poles. $z=1$ is a pole of order 1, $z=2$ is a pole of order 2. We compute

$$\operatorname{res}_1 f = \lim_{z\to 1} (z-1) f(z) = \lim_{z \to 1} \frac{z}{(z-2)^2} = 1$$

$$\operatorname{res}_2 f = \frac{1}{1!} \lim_{z\to 2} \frac{d}{dz}\Big((z-2)^2 f(z) \Big) =$$

$$= \lim_{z\to 2} \left(\frac{z}{z-1} \right)' = \lim_{z \to 2} \frac{1\cdot(z-1) - z\cdot 1}{(z-1)^2} = -1$$

b) Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$.

We compute

$$\operatorname{res}_{\frac{\pi}{2}} f = \operatorname{res}_{\frac{\pi}{2}} \frac{\overbrace{\sin z}^{\varphi}}{\underbrace{\cos z}_{\psi}}$$

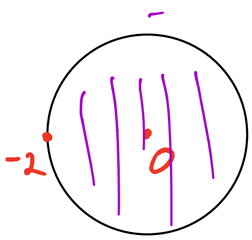
$$\varphi\left(\tfrac{\pi}{2}\right) = 1 \neq 0, \quad \psi\left(\tfrac{\pi}{2}\right) = 0, \quad \psi'\left(\tfrac{\pi}{2}\right) = -\sin\tfrac{\pi}{2} = -1$$

So, $$\operatorname{res}_{\frac{\pi}{2}} f = \frac{\overbrace{\sin\frac{\pi}{2}}^{=\varphi(\frac{\pi}{2})}}{\underbrace{-\sin\frac{\pi}{2}}_{=\psi'(\frac{\pi}{2})}} = -1$$

c) $f(z) = \frac{1}{z+2}\cos\frac{1}{z}$, $a=0$ - essential singularity

We are going to find c_{-1} in the Laurent series for f.

$$f(z)=\frac{1}{z+2}\cos\frac{1}{z} = \frac{1}{2}\cdot\frac{1}{1+\frac{z}{2}}\cos\frac{1}{z} =$$



$$= \frac{1}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\dots\right)\cdot$$

$$\cdot\left(1-\frac{1}{2!\,z^2}+\frac{1}{4!\,z^4}-\dots\right)$$

Hence, $c_{-1} = \frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{2!}-\frac{1}{2^3\,4!}+\frac{1}{2^5\,6!}-\dots+\frac{(-1)^{n-1}}{2^{2n-1}(2n)!}+\dots\right)$.

2. ∞ as an isolated singularity

We recall that ∞ is an isolated singularity of f if f is holomorphic in $\{|z|>R\}$ for some $R>0$. Similarly,

- ∞ is removable if $\lim\limits_{z\to\infty} f(z)$ exists and is finite
- ∞ is a pole if $\lim\limits_{z\to\infty} f(z)=\infty$
- ∞ is an essential singularity, if $\nexists \lim\limits_{z\to\infty} f(z)$.

We remark that ∞ is an isolated singular point of f if and only if 0 is an isolated singular point of

$$\tilde{f}(z) = f\left(\frac{1}{z}\right).$$

We define the Laurent expansion of f at infinity as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \qquad (23.1)$$

where the series converges for $R < |z| < \infty$.

Next, we characterize the type of singularity at ∞ via the Laurent expancion.

We write the Laurent series of $\tilde{f}$ at 0:

$$\tilde{f}(z) = \sum_{n=-\infty}^{+\infty} \tilde{c}_n z^n.$$

Hence,

$$f(z) = \tilde{f}\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{+\infty} \tilde{c}_n z^{-n} =$$

(23.2)

$$= \sum_{n=-\infty}^{\infty} \tilde{c}_{-n} z^n = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where $c_n = \tilde{c}_{-n}$.

We will call $\sum_{n=-\infty}^{0} c_n z^n$ the regular part of the Laurent series, and

$\sum_{n=1}^{\infty} c_n z^n$ its principal part.

The equality (23.2) immediately implies that ∞ is

- removable if the principal part of (23.1) equals 0
- a pole if the principal part has a finite number of nonzero terms
- an essential singularity if the principal part consists of an infinite number of nonzero terms.

Def 23.1 Let infinity be an isolated singularity of the function f. The residue of f at infinity is

$$\operatorname{res}_{\infty} f = \frac{1}{2\pi i} \int_{\gamma_\rho^-} f(z)\,dz,$$

where γ_ρ^- is the circle $\{|z| = \rho\}$ of a sufficiently large radius R oriented clockwise.

According to the Laurent Theorem 21.4,

$$\boxed{\operatorname{res}_{\infty} f = -c_{-1}}$$

Theorem 23.1 Let the function f be holomorphic in the complex plane $\mathbb{C}$ except at a finite number of points $a_1, \dots, a_n$. Then

$$\sum_{k=1}^{n} \mathrm{res}_{a_k} f + \mathrm{res}_\infty f = 0.$$

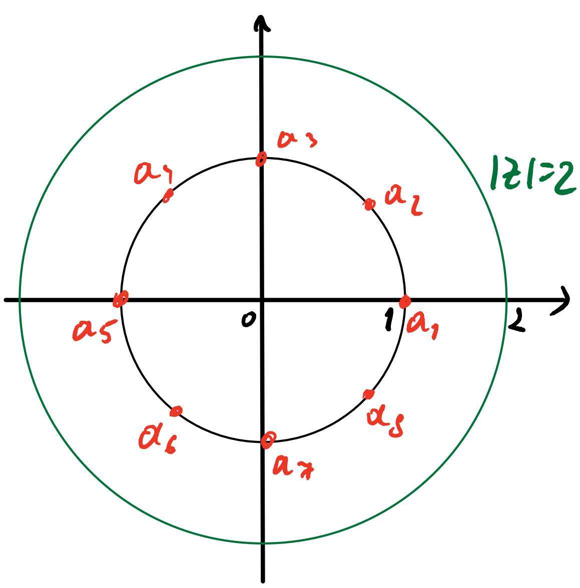
Proof Let the circle $\gamma_\rho = \{ |z| = \rho \}$ surrounds all $a_1, \dots, a_n$. Then by Theorem 22.7,

$$\underbrace{\frac{1}{2\pi i} \int_{\gamma_\rho} f(z)\, dz}_{= -\mathrm{res}_\infty f} = \sum_{k=1}^{n} \mathrm{res}_{a_k} f .$$

∎

Ex 23.2 We compute for $f = \frac{1}{(z^8+1)^2}$

$$\int_{|z|=2} \frac{dz}{(z^8+1)^2} = 2\pi i \sum_{k=1}^{8} \mathrm{res}_{a_k} f = -2\pi i\, \mathrm{res}_\infty f.$$



We compute c_{-1} in the Laurent series at ∞:

$$\frac{1}{(z^8+1)^2} = \frac{1}{z^{16}} \cdot \frac{1}{\left(1 + \frac{1}{z^8}\right)^2} =$$

$$= \frac{1}{z^{16}} \, \frac{1}{1 + \frac{1}{z^8}} \cdot \frac{1}{1 + \frac{1}{z^8}} =$$

$$= \frac{1}{z^{16}} \left(1 - \frac{1}{z^8} + \dots \right)\left(1 - \frac{1}{z^8} + \dots \right).$$

Hence $c_{-1} = 0$. So,

$$\int_{|z|=2} \frac{dz}{(z^8+1)^2} = 0.$$

3. Application to Riemann integrals

a) $\displaystyle\int_0^{2\pi} R(\cos\varphi, \sin\varphi)\, d\varphi$, where

$$R(t,s) = \frac{P(t,s)}{Q(t,s)}$$ and P, Q are polynomials of t, s.

We recall that $z = e^{i\varphi} = \cos\varphi + i\sin\varphi$

$\bar{z} = e^{-i\varphi} = \cos\varphi - i\sin\varphi = \frac{1}{z}$

Hence, $\cos\varphi = \dfrac{z+\bar{z}}{2}$, $\sin\varphi = \dfrac{z-\bar{z}}{2i}$

If $\varphi \in [0, 2\pi]$, then $z = e^{i\varphi}$ defines the circle $|z|=1$. Next, we compute $d\varphi$

$$dz = d\, e^{i\varphi} = i e^{i\varphi} d\varphi = iz\, d\varphi$$

So,

$$d\varphi = \frac{dz}{iz}.$$

So,

$$\int_0^{2\pi} R(\cos\varphi, \sin\varphi)\, d\varphi \overset{z=e^{i\varphi}}{=} \int_{|z|=1} R\left(\frac{z+\overset{=\frac{1}{z}}{\bar{z}}}{2}, \frac{z-\overset{=\frac{1}{z}}{\bar{z}}}{2i}\right)\frac{dz}{iz}$$

Ex 23.3

$$\int_0^{2\pi} \frac{d\varphi}{5-4\cos\varphi} = \int_{|z|=1} \frac{dz}{iz\left(5-4\,\frac{z+\bar{z}}{2}\right)} =$$

$z = e^{i\varphi}$

$d\varphi = \frac{dz}{iz}$

$\cos\varphi = \frac{z+\bar{z}}{2}$

$a_1 = 2$, $a_2 = \frac{1}{2}$

are isol. sing. points

$$= \frac{1}{i} \int\limits_{|z|=1} \frac{dz}{5z - 2z^2 - 4\underbrace{\bar{z}z}_{=1}} =$$

$$= -\frac{1}{i} \int\limits_{|z|=1} \frac{dz}{2z^2 - 5z + 2} =$$

$$= -\frac{1}{i} 2\pi i \, \operatorname{res}_{\frac{1}{2}} \frac{1}{2z^2 - 5z + 2} =$$

$$= -2\pi \frac{1}{(2z^2 - 5z + 2)'} \Big|_{z=\frac{1}{2}} =$$

$$= \frac{2\pi}{(4z - 5)|_{z=\frac{1}{2}}} = \frac{-2\pi}{2-5} = \frac{2}{3}\pi .$$

6) $\int\limits_{-\infty}^{+\infty} R(x)\,dx$, where $R(t) = \frac{P(t)}{Q(t)}$,

P, Q are polynomials, $\deg P \leq \deg Q - 2$

and $Q(t) \neq 0 \;\; \forall\, t \in \mathbb{R}$. Then

$$\int\limits_{-\infty}^{+\infty} R(x) = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{a_k} R,$$

where a_k are zeros of Q such that

$\operatorname{Im} a_k > 0$

Ex 23.4 We compute

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^3} = 2\pi i \operatorname{res}_i \frac{1}{(1+z^2)^3} =$$

$z^2+1=0$

$z^2=-1$

$z = \pm i$

$\uparrow$

poles of order 3

$$= 2\pi i \frac{1}{2!} \lim_{z\to i} \left(\frac{(z-i)^3}{(z^2+1)^3} \right)'' =$$

$$= \pi i \lim_{z\to i} \left(\frac{1}{(z+i)^3} \right)'' =$$

$$= \pi i \lim_{z\to i} (-3)(-4)(z+i)^{-5} =$$

$$= \pi i \lim_{z\to i} \frac{12}{(z+i)^5} = \pi i \frac{12}{(2i)^5} = \frac{\pi i \cdot 12}{i \cdot 32} =$$

$$= \frac{3}{8}\pi .$$