22. Residues

1. Isolated singular points

In this section, we will study points where analyticity of a function is violated.

Def 22.1 A point $a \in \bar{\mathbb{C}}$ is an *isolated singular point* of a function f if there exists a punctured neighborhood of this point (that is, a set of the form $0<|z-a|<r$ if $a \neq \infty$, or $R<|z|<\infty$ if $a=\infty$), where f is holomorphic.

We distinguish three types of singular points depending on the behavior of f near such point.

Def 22.2 An isolated singular point a of a function f is said to be

a) *removable* if the limit $\lim\limits_{z\to a} f(z)$ exists and is finite;

b) a *pole* if the limit $\lim\limits_{z\to 0} f(z)$ exists and is equal to ∞

c) an *essential singularity* if f has neither a finite nor infinite limit as $z\to a$.

Example 22.1 a) The function $f(z)=\dfrac{\sin z}{z}$

has removable singularity, since
$$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{4!} - \dots$$
and $\lim_{z \to 0} f(z) = 1$.

b) The function $f(z) = \frac{1}{z^n}$, where $n \in \mathbb{N}$, has a pole at $z = 0$.

c) The function $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$. Indeed, if $z = x \in \mathbb{R}$ then the limit of f as x tends to 0 from the left and right are different:
$$\lim_{x \to 0+} e^{\frac{1}{x}} = +\infty$$
$$\lim_{x \to 0-} e^{\frac{1}{x}} = 0.$$

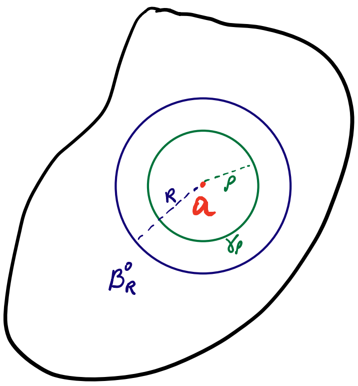
Also it has no limit as z goes to zero along the imaginary axis:
$$\lim_{y \to 0} e^{iy} = \lim_{y \to 0} \left(\cos \frac{1}{y} + i \sin \frac{1}{y}\right)$$
does not exists.

Theorem 22.1 An isolated singular point $a \in \mathbb{C}$ of a function f is a removable singularity iff its Laurent expansion
$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n$$

contains no principal part, i.e.

$$c_{-1} = c_{-2} = \dots = 0.$$

Proof Let a be a removable singularity of f, then the limit $\lim\limits_{z\to a} f(z) = A$ exists and is finite. This implies that f is bounded in a punctured neighborhood $\{0<|z-a|<R\} =: B_R^\circ$ of f. Say, $|f| < M$. By Laurent theorem (see Theorem 21.4),



$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta-a)^{n+1}}, \quad n = 0, \pm 1, \pm 2, \dots$$

We estimate for $n = -1, -2, \dots$

$$|c_n| = \frac{1}{2\pi} \left| \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta-a)^{n+1}} \right| \le \frac{1}{2\pi} \int_{\gamma_\rho} \frac{|f(\zeta)|}{\rho^{n+1}}\, ds \le$$

$$\le \frac{1}{2\pi} \frac{M}{\rho^{n+1}} \cdot \text{length}\, \gamma_\rho = \frac{1}{2\pi} \frac{M}{\rho^{n+1}} 2\pi\rho = \frac{M}{\rho^n} =$$

$$= M\rho^{-n} \longrightarrow 0, \quad \rho \to 0$$

So, $c_{-1} = c_{-2} = \dots = 0$ ∎

The similar argument gives the following statement

Theorem 22.2 An isolated singular point a of a function f is removable iff f is bounded in a neighborhood of the point a.

Theorem 22.3 An isolated singular point $a \in \mathbb{C}$ is a pole of f if its Laurent expansion near a has a form

$$f(z) = \sum_{n=-N}^{\infty} c_n (z-a)^n \qquad (22.1)$$

for some $N \in \mathbb{N}$, and $c_N \neq 0$.

Def 22.3 The number N in (22.1) is called the order of a pole of f.

We note another simple fact that relates poles and zeros.

Theorem 22.4 A pole a is a pole of the function f iff the function $\varphi = \frac{1}{f}$ is holomorphic in a neighborhood of a and $\varphi(a) = 0$.

In the next section, we will need to compute the order of a pole.

Def 22.4 The order of a zero $a \in \mathbb{C}$ of a function φ holomorphic at this point, is the order of the first non-zero derivative $\varphi^{(k)}(a)$.

Proposition 22.1 The order of the pole a of a function f is the order of this point as a zero of $\varphi = \frac{1}{f}$.

Now we give the characterization of an essential singularity.

Theorem 22.5 An isolated singular point a of f is an essential singularity iff the principal part of the Laurent expansion of f near a

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

contains infinitely many non-zero terms.

At the end we give interesting property of essential singularity

Theorem 22.6 If a is an essential singularity of a function f then for any $A \in \overline{\mathbb{C}}$ we may find a sequence $\{z_n\}_{n\geq 1}$ such that

$$z_n \to a, \quad n \to \infty,$$

and
$$\lim_{n\to\infty} f(z_n) = A.$$

2. Residues

Def 22.5 Let $a \in \mathbb{C}$ be an isolated singular point of f. The number

$$\operatorname{res}_a f = \frac{1}{2\pi i} \int_{\gamma_\rho} f(z)\, dz$$

$\gamma_\rho \leftarrow$ positively oriented

is called the residue of f at a, where $\gamma_\rho = \{z : |z-a| = \rho\}$, $0 < \rho < R$, and f is holomorphic in $0 < |z-a| < R$.

Proposition 22.2 The residue of a function f at an isolated singular point $a \in \mathbb{C}$ is equal to the coefficient in front of the term $(z-a)^{-1}$ in its Laurent expansion around a:

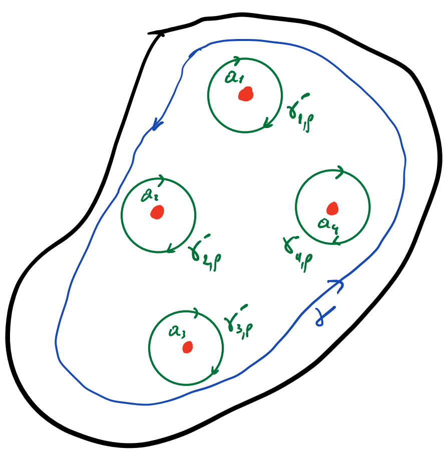
$$\boxed{\operatorname{res}_a f = c_{-1}} \qquad (22.2)$$

Proof The equality (22.2) follows from the formula for c_{-1}. (see Theorem 21.4) ∎

Theorem 22.7 Let the function f be holomorphic everywhere in a domain U (open connected subset of $\mathbb{C}$) except at an isolated set of singular points $a_1, \dots, a_n$. Let γ be a positively oriented simple connected path in U surrounding $a_1, \dots, a_n$. Then

$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{a_k} f}$$

Proof. The statement follows from the Proposition 18.3. Indeed



$$\int_\gamma f(z)\,dz + \sum_{k=1}^{n} \int_{\gamma_{k,\rho}^-} f(z)\,dz = 0$$

Hence,

$$\int_\gamma f(z)\,dz = -\sum_{k=1}^{n} \int_{\gamma_{k,\rho}^-} f(z)\,dz =$$

$$= 2\pi i \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_{k,\rho}} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{a_k} f.$$