19 The Taylor series

1. Uniform converges of serieses

Let a_n, $n \geq 1$, be complex numbers. We recall that a series

$$\sum_{n=0}^{\infty} a_n$$

is convergent if the sequence of its partitial sums

$$S_n = \sum_{k=0}^{n} a_k$$

has a finite limit S. This limit is called the sum of the series.

A functional series

$$\sum_{n=0}^{\infty} f_n(z)$$

with the functions f_n defined on a set $M \subseteq \overline{\mathbb{C}}$ converges uniformly on M if it converges at all $z \in M$, and, moreover,

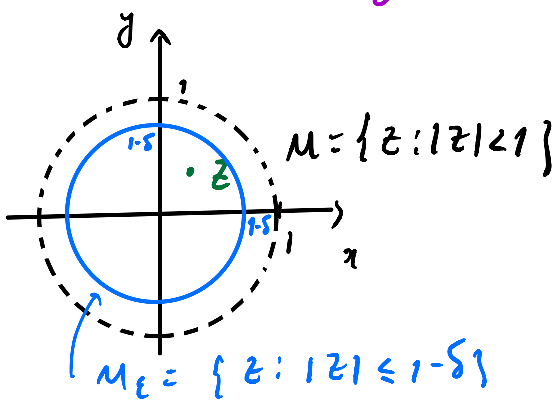
$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N$$

$$\left| \sum_{k=n+1}^{\infty} f_k(z) \right| = \left| f(z) - \sum_{k=0}^{n} f_k(z) \right| < \varepsilon, \quad \forall z \in M,$$

where
$$f(z) = \sum_{n=0}^{\infty} f_n(z).$$

Example 18.1 Consider the series

$$\sum_{n=0}^{\infty} z^n, \quad |z| < 1 \qquad (19.1)$$



We remark that series (19.1) converges for every $z \in M$. Indeed,

$$S_n = \sum_{k=0}^{n} z^k = 1 + z + z^2 + \dots + z^n$$

$$-$$

$$z S_n = z + z^2 + \dots + z^{n+1}$$

$$S_n (1 - z) = 1 - z^{n+1}$$

$$S_n = \frac{1 - z^{n+1}}{1 - z} = \frac{1 - r^{n+1}(\cos(n+1)\varphi + i \sin(n+1)\varphi)}{1 - z}$$

$$\longrightarrow \frac{1}{1 - z}, \quad \text{where } z = r(\cos\varphi + i \sin\varphi).$$

But series (19.1) converges uniformly only on M_δ for any $\delta > 0$.

Indeed,

$$\left| \sum_{k=n+1}^{\infty} z^k \right| = \left| \frac{1}{1-z} - \frac{1 - z^{n+1}}{1 - z} \right| =$$

$$= \frac{|z^{n+1}|}{|1-z|} = \frac{|z|^{n+1}}{|1-z|} \le \frac{(1-\delta)^{n+1}}{\delta} \xrightarrow[n\to\infty]{} 0,$$

$$\forall x \in M_\delta.$$

So, (18.1) converges uniformly on M_δ.

Assume that (19.1) converges uniformly on M, then for any $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n \ge N$

$$\left| \sum_{k=n+1}^{\infty} z^k \right| = \frac{|z|^{n+1}}{|1-z|} < \varepsilon \quad \forall |z| < 1$$

Take $z = x = x + i0$, $x > 0$. Then

$$\left| \sum_{k=n+1}^{\infty} z^k \right| = \frac{x^{n+1}}{1-x} < \varepsilon \qquad (19.2)$$

$\to +\infty$ if $x \to 1-$

So inequality (19.2) does not hold for x closed to 1. Consequently, (19.1) does not converges uniformly on

$$M = \{ z : |z| < 1 \}$$

Exercise 18.1 Show that the series

$$\sum_{n=0}^{\infty} f_n(z) \qquad (19.3)$$

converges uniformly on M if the series

$$\sum_{n=0}^{\infty} \|f_n\|$$

converges, where $\|f_n\| = \sup_{z \in M} |f_n(z)|$.

Solution Let $S(z) = \sum_{n=0}^{\infty} f_n(z)$, $S_n(z) = \sum_{k=0}^{n} f_k(z)$.

We first estimate for $m > n$

$$|S_m(z) - S_n(z)| = \left| \sum_{k=n+1}^{m} f_k(z) \right| \leq$$

$$\leq \sum_{k=n+1}^{m} |f_k(z)| \leq \sum_{k=n+1}^{m} \|f_k\| \quad \forall z \in M.$$

We fix $\varepsilon > 0$.

Since the series $\sum_{n=0}^{\infty} \|f_n\|$ converges, there exists $N \in \mathbb{N}$ s.t. $\forall n, m \geq N$, $n < m$

$$\sum_{k=n+1}^{m} \|f_k\| < \frac{\varepsilon}{2},$$

by the Cauchy criterion (see Th 19.3 Math I).

So,

$$|S_m(z) - S_n(z)| < \frac{\varepsilon}{2} \quad \forall z \in M,$$

$$\forall n, m \geq N_0.$$

Making $m \to \infty$, we have

$$|S(z) - S_n(z)| \leq \frac{\varepsilon}{2} < \varepsilon \quad \forall z \in M \quad \forall n \geq N_0.$$

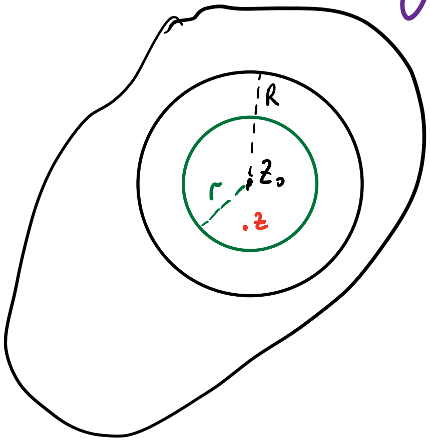
This implies the uniform converges of (18.3).

2. The Taylor series

Theorem 19.1 Let f be holomorphic in U and $z_0 \in U$. Then the function f may be represented as a sum

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

inside any disk $B_R = \{|z - z_0| < R\} \subset U$



Proof Let $z \in B_R$ be an arbitrary point. Choose $r > 0$ so that $|z - z_0| < r < R$ and denote by $\gamma_r = \{\zeta : |\zeta - z_0| = r\}$

The integral Cauchy formula implies that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

We write

$$(19.4) \quad \frac{1}{\zeta - z} = \left[(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right) \right]^{-1} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$

We multiply both sides by $\frac{1}{2\pi i} f(\zeta)$ and integrate the series term-wise along γ_r. The series (19.4) converges uniformly, since

$$\left| \frac{z-z_0}{\zeta - z_0} \right| = \frac{|z-z_0|}{r} = q < 1,$$

and

$$\sum_{n=0}^{\infty} q^n < \infty.$$

Consequently, the term-wise integration is legitimate and we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \sum_{n=0}^{\infty} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}} (z-z_0)^n =$$

$$= \sum_{n=0}^{\infty} c_n (z-z_0)^n,$$

where

$$\boxed{c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}, \quad n = 0, 1, 2, \ldots}$$

■

Def 19.1 The power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n, \qquad (19.5)$$

where

$$c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}, \qquad (19.6)$$

is the Taylor series of the function f at the point z_0 (or centered at z_0)

The Cauchy inequality Let the function f be holomorphic in a closed disk

$$\overline{B}_r = \{ |z - z_0| \le r \}$$

and let its absolute value on the circle $\gamma_r = \partial U$ be bounded by a constant M. Then

$$|c_n| \le \frac{M}{r^n}, \quad n = 0, 1, \ldots$$

Proof From (19.6) we have

$$|c_n| \le \frac{1}{2\pi} \int_{\gamma_r} \frac{|f(\zeta)|\, ds}{|\zeta - z_0|^{n+1}} = \frac{1}{2\pi i} \frac{M}{r^{n+1}} 2\pi r =$$

$$= \frac{M}{r^n}.$$

Theorem 19.2 (Liouville) If the function f is holomorphic in the whole complex plane and bounded then it is equal identically to a constant

Proof According to Th 19.1, the function f may be represented by a Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

in any closed disk $\overline{B}_R = \{|z| \le R\}$, $R < \infty$, with the coeddicients that do not depend on R. Since f is bounded on $\mathbb{C}$, we have

$$|f(z)| \le M, \quad \forall z \in \mathbb{C}.$$

By Cauchy inequalities $\forall n = 0, 1, 2, \ldots$

$$|c_n| \le \frac{M}{R^n} \to 0, \quad R \to \infty.$$

Therefore $c_1 = c_2 = \ldots = 0$ and

$$f(z) = c_0.$$

∎