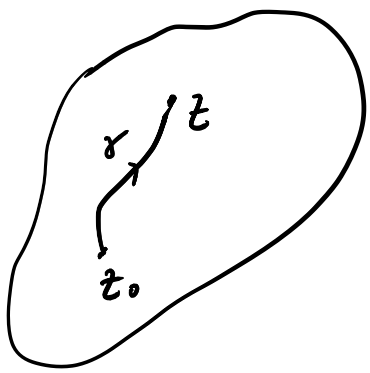
18. Cauchy integral formula

1. Consequences of Cauchy's theorem

Proposition 18.1 Any function f holomorphic in a simply connected domain U has an antiderivative in this domain.

Proof Let $z \in U$ and $z_0 \in U$ be a fixed point. We remark that the integral over any piecewise continuously differentiable path γ joining z_0 and z only depends on z_0 and z, by Cauchy's theorem. So, we will use the notation

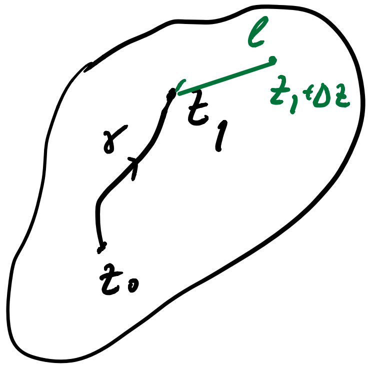


$$\int_\gamma f(z)\,dz = \int_{z_0}^{z} f(z)\,dz$$

We define a function $F: U \to \mathbb{C}$ as follows

$$F(z) = \int_{z_0}^{z} f(\xi)\,d\xi.$$

Let us show that F is an antiderivative of f. By the definition of complex derivative (see Def 14.1),



$$F'(z_1) = \lim_{\Delta z \to 0} \frac{F(z_1+\Delta z) - F(z_1)}{\Delta z}$$

Let ℓ be a path connected z_1 and $z_1 + \Delta z$.

We estimate

$$\left| \frac{F(z_1+\Delta z) - F(z_1)}{\Delta z} - f(z_1) \right| =$$

$$= \frac{1}{\Delta z} \left| \int_{z_0}^{z_1+\Delta z} f(\zeta)\,d\zeta - \int_{z_0}^{z_1} f(\zeta)\,d\zeta - f(z_1)\Delta z \right| =$$

$$= \frac{1}{\Delta z} \left| \int_{z_1}^{z_1+\Delta z} f(\zeta)\,d\zeta - \int_{z_1}^{z_1+\Delta z} f(z_1)\,d\zeta \right| =$$

$$= \frac{1}{\Delta z} \left| \int_{z_1}^{z_1+\Delta z} \big(f(\zeta) - f(z_1)\big)\,d\zeta \right| \underset{\text{Lecture 17, Sec. 2 prop. 5)}}{\leq}$$

$$\leq \frac{1}{\Delta z} \max_{\zeta \in \ell} |f(\zeta) - f(z_1)| \underbrace{\text{length}(\ell)}_{= \Delta z} =$$

$$= \max_{\zeta \in \ell} |f(\zeta) - f(z_1)| \to 0$$

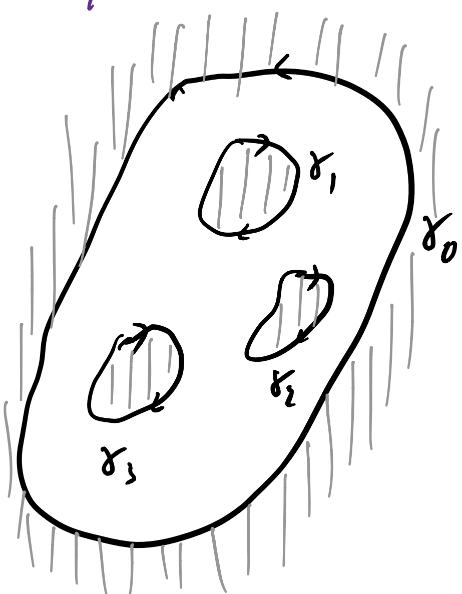
Since f is continuous according to Prop. 14.1.

We will say that f is holomorphic on $\overline{U}$ if f may be extended to into a domain G that contains D.

Proposition 18.2 (Generalization of Cauchy's theorem)
Let f be holomorphic in $\overline{U}$. Let also ∂U be piecewise continuously differentiable curve. Then

$$\int_{\partial G} f(z)\,dz = 0.$$

Def 18.1 Let the boundary of a bounded domain U consists of a finite number of closed curves γ_k, $k=0,1,2,\dots,n$. which are positively oriented. The boundary with this orientation is called the **oriented boundary** and is denoted by ∂U.

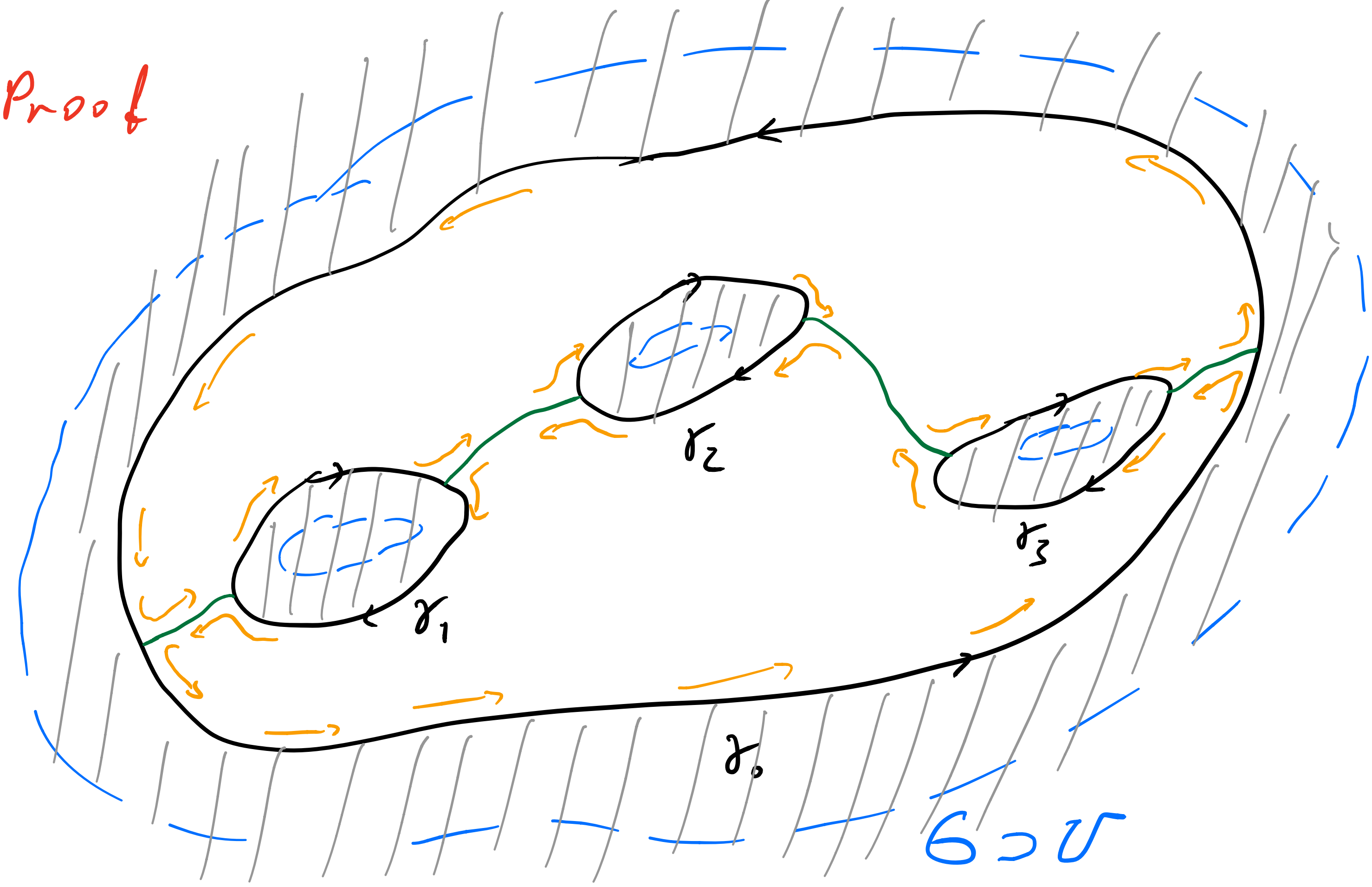


Proposition 18.3 Let a bounded domain be bounded by a finite number of piecewise continuously differentiable curves and let f be holomorphic in its closure $\overline{U}$. Then

$$\int_{\partial U} f(z)\,dz = \int_{\gamma_0} f\,dz + \sum_{k=1}^{n} \int_{\gamma_k} f\,dz = 0$$

$\leftarrow$ oriented boundary

Proof



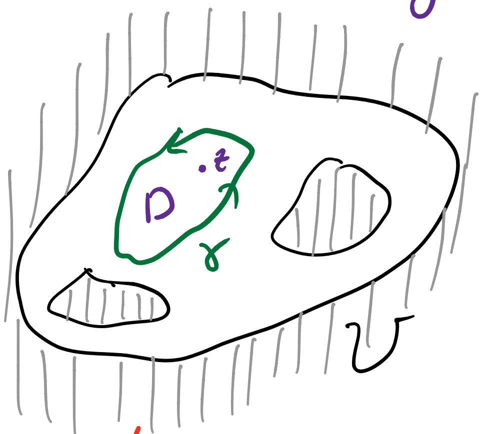
The proposition immediately follows from Cauchy's theorem. ∎

2. The Cauchy integral formula

Theorem 18.1 Let a function f in $\bar{U}$, where U is a bounded domain bounded by a finite number of piecewise continuously differentiable curves. Then for every $z \in U$

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\xi)}{\xi - z}\, d\xi .$$

Consequence 18.1 Let f be holomorphic in U, where U be an open set. Let γ be a simple continuously diff. curve in U surrounding a set D contained in U.

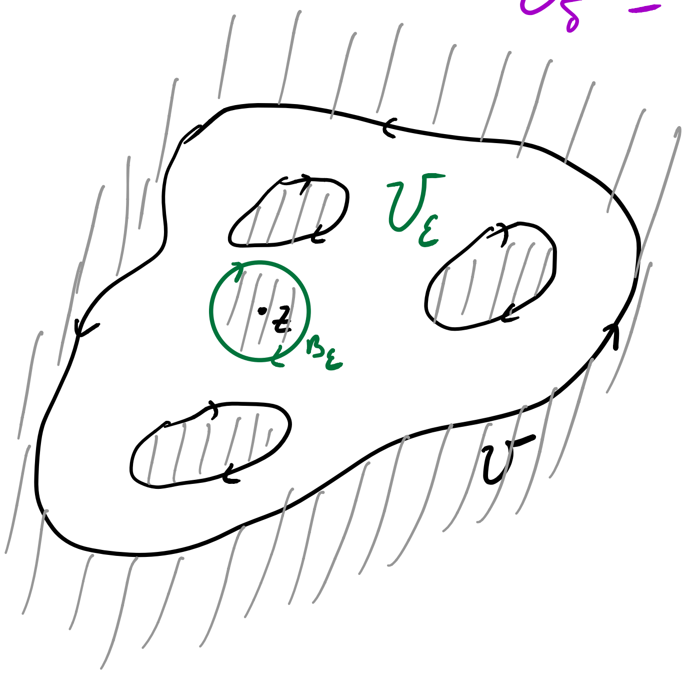


Then for any $z \in D$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Proof Let $z \in U$ and $\varepsilon > 0$ be such that the disk $B_\delta = \{\zeta \in \mathbb{C} : |\zeta - z| < \delta\}$ is contained in U and

$$U_\delta = U \setminus B_\delta.$$



Remark that the oriented boundary of U_δ consists of ∂U and

$$\partial B_\delta = \{\zeta \in \mathbb{C} : |\zeta - z| = \delta\}$$

oriented clockwise.

Moreover, f is holomorphic in U_δ. So, by Prop. 18.3

$$\int_{\partial U_\delta} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{\partial B_\delta} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 0$$

Hence
$$\int_{|\zeta - z| = \delta} \frac{f(\zeta)}{\zeta - z} = \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, dz$$

↑ oriented counter clockwise

Next, we consider

$$\left| f(z) - \frac{1}{2\pi i} \int_{|\zeta - z| = \delta} \frac{f(\zeta)}{\zeta - z} d\zeta \right| =$$

$$= \left| \frac{1}{2\pi i} f(z) \underbrace{\int_{|\zeta - z| = \delta} \frac{d\zeta}{\zeta - z}}_{= 2\pi i \text{ (see Ex 17.1)}} - \frac{1}{2\pi i} \int_{|\zeta - z| = \delta} \frac{f(\zeta)}{\zeta - z} d\zeta \right| =$$

$$\leq \frac{1}{2\pi} \int_{|\zeta - z| = \delta} \frac{|f(z) - f(\zeta)|}{\underbrace{|\zeta - z|}_{= \delta}} ds \leq$$

[see Lect. 17, section 2, prop 5)

$$\leq \frac{1}{2\pi} \frac{1}{\delta} \max_{\partial B_\delta} |f(z) - f(\zeta)| \underbrace{\text{length}(\partial B_\delta)}_{= 2\pi\delta}$$

$$\to 0, \quad \delta \to 0,$$

since the function f is continuous.

$\blacksquare$

3. The Taylor series

Let a_n, $n \geq 1$, be complex numbers. We say that a series

$$\sum_{n=0}^{\infty} a_n$$

is convergent if the sequence of its partitial sums

$$S_n = \sum_{k=0}^{n} a_k$$

has a finite limit S. This limit is called the sum of the series.

A functional series

$$\sum_{n=0}^{\infty} f_n(z)$$

with the functions f_n defined on a set $M \subseteq \overline{\mathbb{C}}$ converges uniformly on M if it converges at all $z \in M$, and, moreover,

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N$$

$$\left| \sum_{k=n+1}^{\infty} f_k(z) \right| = \left| f(z) - \sum_{k=0}^{n} f_k(z) \right| < \varepsilon, \ \forall z \in M,$$

where $$f(z) = \sum_{n=0}^{\infty} f_n(z).$$