17. Cauchy's theorem

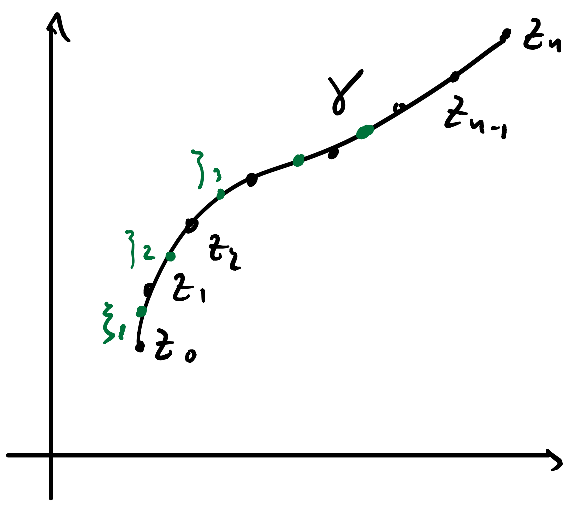
1. Integration

Let U be an open set in $\mathbb{C}$ and γ is piecewise continuously differentiable path in U. Let $f: \gamma \to \mathbb{C}$ be a continuous function

Def 17.1 If there exists

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k) \Delta z_k =: \int_\gamma f(z)\, dz,$$

where $\lambda = \max |\Delta z_k|$, $\Delta z_k = z_k - z_{k-1}$, that does not depend on choice of a partition $\{z_k\}$ and points $\{\xi_k\}$, then the limit is called *integrable of f along the path γ*.



We next state a connection of the integral with the line integral.
We rewrite the integral sum.

So, $\Delta z_k = z_k - z_{k-1} = (x_k - x_{k-1}) + i(y_k - y_{k-1}) =$
$= \Delta x_k + i \Delta y_k$

and

$f(z) = f(x+iy) = u(x,y) + i v(x,y)$.

Let also $\xi_k = \eta_k + i\zeta_k$. Then

$$\sum f(\xi_k)\Delta z_k = \sum_{k=1}^{n} \left(u(\eta_k, \zeta_k) + i v(\eta_k, \zeta_k)\right) \cdot$$
$$\cdot (\Delta x_k + i \Delta y_k) =$$
$$= \sum_{k=1}^{n} \left(u(\eta_k, \zeta_k)\Delta x_k - v(\eta_k, \zeta_k)\Delta y_k\right) +$$
$$+ i \sum_{k=1}^{n} \left(v(\eta_k, \zeta_k)\Delta x_k + u(\eta_k, \zeta_k)\Delta y_k\right)$$

This immediately implies

$$\int_\gamma f(z)\,dz = \int_\gamma u(x,y)\,dx - v(x,y)\,dy +$$
$$+ i \int_\gamma v(x,y)\,dx + u(x,y)\,dy$$

2. Properties of the integral

1) $\int_\gamma (\alpha f + \beta g)\,dz = \alpha \int_\gamma f\,dz + \beta \int_\gamma g\,dz$, where $\alpha, \beta \in \mathbb{C}$.

2) $\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz$, where γ^- is obtained from γ by a change of orientation

3) $\int_{\gamma_1 \cup \gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$,

where the end point of γ_1 is the initial point of γ_2

4) $\left| \int_\gamma f(z)\, dz \right| \le \int_\gamma |f(z)|\, ds$ ← line integral of the scalar field.

it follows from the inequality

$$\left| \sum_{k=1}^{n} f(\zeta_k) \Delta z_k \right| \le \sum_{k=1}^{n} |f(\zeta_k)| \cdot |\Delta z_k| \le$$

$$\le \sum_{k=1}^{n} |f(\zeta_k)| \underbrace{\Delta s_k}_{\text{length of } \gamma \text{ between } z_{k-1}, z_k.}$$

5) $\left| \int_\gamma f(z)\, dz \right| \le \max_{z\in\gamma} |f(z)| \cdot \text{length}\, \gamma.$

It follows from 4).

We next obtain a formula for computation of the integral.

Let $\gamma(t) = x(t) + i y(t)$, $t \in [\alpha, \beta]$.

Then $\int_\gamma f(z)\, dz = \int_\gamma u\, dx - v\, dy + i \int_\gamma v\, dx + u\, dy$

$$= \int_\alpha^\beta \Big(u(x(t), y(t))\, x'(t) - v(x(t), y(t))\, y'(t) \Big)\, dt$$

$$+ i \int_{\alpha}^{\beta} \left(v(x(t), y(t))\, x'(t) + u(x(t), y(t))\, y'(t) \right) dt$$

$$= \int_{\alpha}^{\beta} \left(u(x(t), y(t)) + i\, v(x(t), y(t)) \right) \left(x'(t) + i y'(t) \right) dt$$

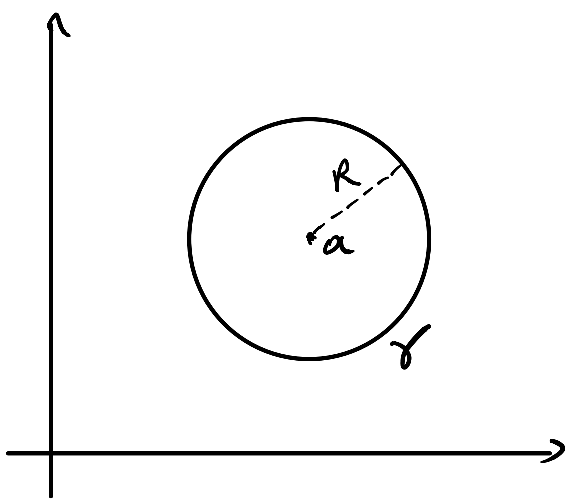
$$= \int_{\alpha}^{\beta} f(\gamma(t))\, \gamma'(t)\, dt.$$

Consequently, we have obtained

$$\int_{\gamma} f(z)\, dz = \int_{\alpha}^{\beta} f(\gamma(t)) \cdot \gamma'(t)\, dt,$$

where $\gamma = \gamma(t)$, $t \in [\alpha, \beta]$.

Ex 17.1 Let $\gamma(t) = a + Re^{it}$, $t \in [0, 2\pi]$



We compute

$$\int_{|z-a|=R} \frac{dz}{(z-a)^n} = \int_{\gamma} \frac{dz}{(z-a)^n},$$

$n \in \mathbb{Z}$.

integral does not depend on the parametrization

Let $n \neq 1$, then

$$\int_{|z-a|=R} \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{1}{R^n e^{int}} \cdot iRe^{it}\, dt =$$

$$= \frac{i}{R^{n-1}} \int_0^{2\pi} e^{-i(n-1)t}\, dt = -\frac{1}{R^{n-1}} \frac{1}{i(n-1)} e^{-i(n-1)t}\Big|_0^{2\pi}$$

$$= -\frac{1}{R^{n-1}} \cdot \frac{1}{i(n-1)} \Big(\cos(n-1)t - i\sin(n-1)t\Big)\Big|_0^{2\pi}$$

$$= 0$$

if $n=1$, then

$$\int_{|z-a|=R} \frac{dz}{z-a} = i\int_0^{2\pi} dt = 2\pi i$$

Hence

$$\int_{|z-a|=R} \frac{dz}{(z-a)^n} = \begin{cases} 0, & n \neq 1 \\ 2\pi i, & n=1. \end{cases}$$

3. Cauchy's Theorem

We start from the following proposition

Proposition 17.1 Let $f: U \to \mathbb{C}$ be continuous, and suppose that f has an antiderivative F which is holomorphic on U $(F' = f)$.

If γ is any path piecewise continuously differentiable path joining z_1 and z_2 from U, then

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1).$$

In particular, if γ is a closed path in U, then

$$\int_\gamma f(z)\,dz = 0.$$

Proof It is enough to prove the statement for a continuously differentiable curve γ.

So, let $\gamma(t) = (x(t), y(t))$, $t \in [\alpha, \beta]$.

We set

$$h(t) = F(\gamma(t)) = u(x(t), y(t)) + i\,v(x(t), y(t))$$

Than by the chain rule

$$h'(t) = \frac{d}{dt} F(\gamma(t)) = u_x x' + \overbrace{u_y}^{=-v_x} y' +$$

$$+ i\left(v_x x' + \underbrace{v_y}_{=u_x} y'\right) = u_x x' - v_x y' +$$

$$+ i\,(v_x x' + u_x y') = (u_x + i v_x)(x' + i y') =$$

$$= F'(\gamma(t)) \cdot \gamma'(t) = f(\gamma(t)) \cdot \gamma'(t)$$

Consequently,

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt =$$

$$= \int_\alpha^\beta h'(t)\,dt = h(t)\Big|_\alpha^\beta = h(\beta)-h(\alpha) =$$

$$= F(z_2) - F(z_1)$$

Ex 17.2 $\displaystyle\int_{2+3i}^{1-i} z^3\,dz = \frac{z^4}{4}\Big|_{2+3i}^{1-i} = \frac{(1-i)^4}{4} - \frac{(2+3i)^4}{4}$

Theorem 17.1 (Cauchy's Theorem) Let U be a simply connected domain in $\mathbb{C}$ and let $f: U \to \mathbb{C}$ be a holomorphic in U. Assume that γ is a piecewise continuously differentiable path in U joining z_1 and z_2 in U. Then $\int_\gamma f(z)\,dz$ depends only on z_1, z_2 and does not depend on the choice of the path. In particular

$$\oint_\gamma f(z)\,dz = 0$$

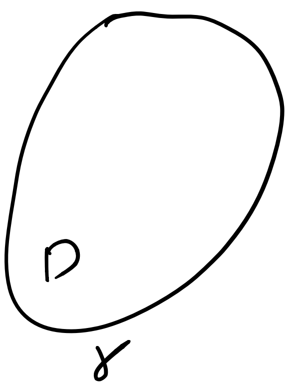
if γ is closed.

Proof We prove the statement under an additional assumption that f' not only exists but is continuous in U.

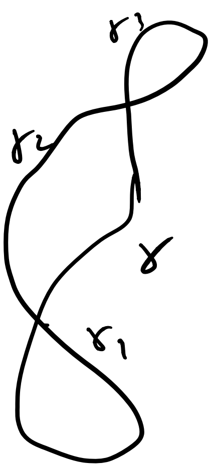
So, let first γ be a closed simple curve (without intersection). Then

$$\oint_\gamma f(z)\,dz = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy =$$

$$= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0,$$



by Green's theorem and Cauchy-Riemann equations.



If γ is any path (with intersections), then we can split it on simple paths.

Next, if z_1, z_2 are two points and γ_1, γ_2 are paths in

γ joining z_1, z_2, then one can take $\gamma = \gamma_1 \cup \gamma_2^-$. This gives

$$0 = \int_{\gamma_1 \cup \gamma_2^-} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2^-} f(z)\,dz =$$

$$= \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz.$$

$\blacksquare$