15. Properties of holomorphic functions.

1. Properties of holomorphic functions.

Let U be an open subset of $\mathbb{C}$.
Let also

$$(15.1) \qquad f(z) = u(x,y) + i\,v(x,y), \quad z = x+iy \in U$$

be a function from U to $\mathbb{C}$.
A function f is called locally constant in U, if for every $z_0 \in U$ $\exists$ a ball $B_r(z_0) \subset U$ s.t. f is constant on $B_r(z_0)$.

We remark that if f is locally constant, then f is constant on every connected component of U.

Lemma 15.1 Let $U \subset \mathbb{C}$ be an open set and $f: U \to \mathbb{C}$ be a holomorphic function on U.
Then

a) If $f'(z) = 0$ $\forall z \in U$, then f is locally constant in U

b) If f only takes real values ($v = 0$) then f is locally constant

c) The functions u, v, defined by (15.1), are harmonic functions, i.e.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

on $\mathcal{U}$.

Proof a) Let $f'(z) = 0 \ \forall z \in \mathcal{U}$. By Th. 14.1

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y) = 0.$$

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0.$

It implies that u, v are locally constant in $\mathcal{U}$.

b) Since $v = 0$, then $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ on $\mathcal{U}$.

By the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$

Hence f is locally constant on $\mathcal{U}$

c) We assume that u, v are twice continuously differentiable.
(Later we will prove that any holomorphic function is complex differentiable infinitely many times. This will implies that u, v are twice continuously diff.)

Computing derivatives of left and right sides of the Cauchy-Riemann equations, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad (15.2)$$

and

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad (15.3)$$

Since $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$, we obtain from (15.2) and (15.3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } U$$

Similarly, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ on U.

Lemma 15.2 Let U be a simply connected domain in $\mathbb{C}$ and $u : U \to \mathbb{R}$ is a harmonic function. Then there exists a holomorphic function $f : U \to \mathbb{C}$ such that

$$u = \operatorname{Re} f.$$

Proof In order to prove the lemma, we find a holomorphic function f satisfying

$$u = \operatorname{Re} f.$$

Let

$$f(z) = u(x,y) + i\, v(x,y), \quad z = x + iy.$$

By the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

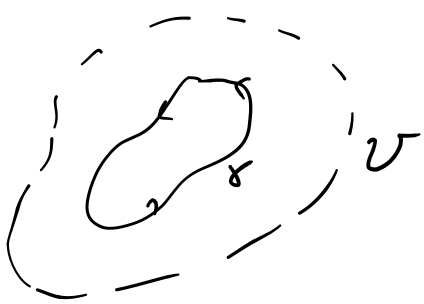
We set

$$\left.\begin{aligned} P &:= \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \\ Q &:= \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \end{aligned}\right\} \quad (15.4)$$

Remark

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } U.$$

If γ is a closed curve in U, then



$$0 = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \int_\Gamma P\,dx + Q\,dy, \text{ by Green's thm.}$$

So, the vector field $\vec{F} = (P, Q)$ is conservative and then potential, according to Prop 9.1. Hence, there exists

$$v : U \to \mathbb{R}$$

such that

$$\nabla v = (P, Q)$$

By (15.4) and Th. 14.1

$$f(z) = u(x,y) + i\, v(x,y)$$

is holomorphic.

2. Some elementary functions

a) The power function

Let $n \in \mathbb{N}$. The function

$$f(z) = z^n, \quad z \in \mathbb{C}$$

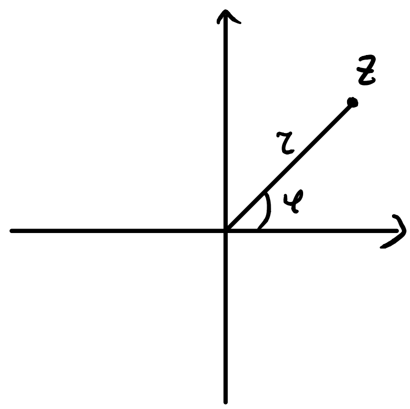
is holomorphic in $\mathbb{C}$. This follows from Prop 14.2 a). Its derivative is

$$f'(z) = n\, z^{n-1}.$$

If we write z in the polar coordinates

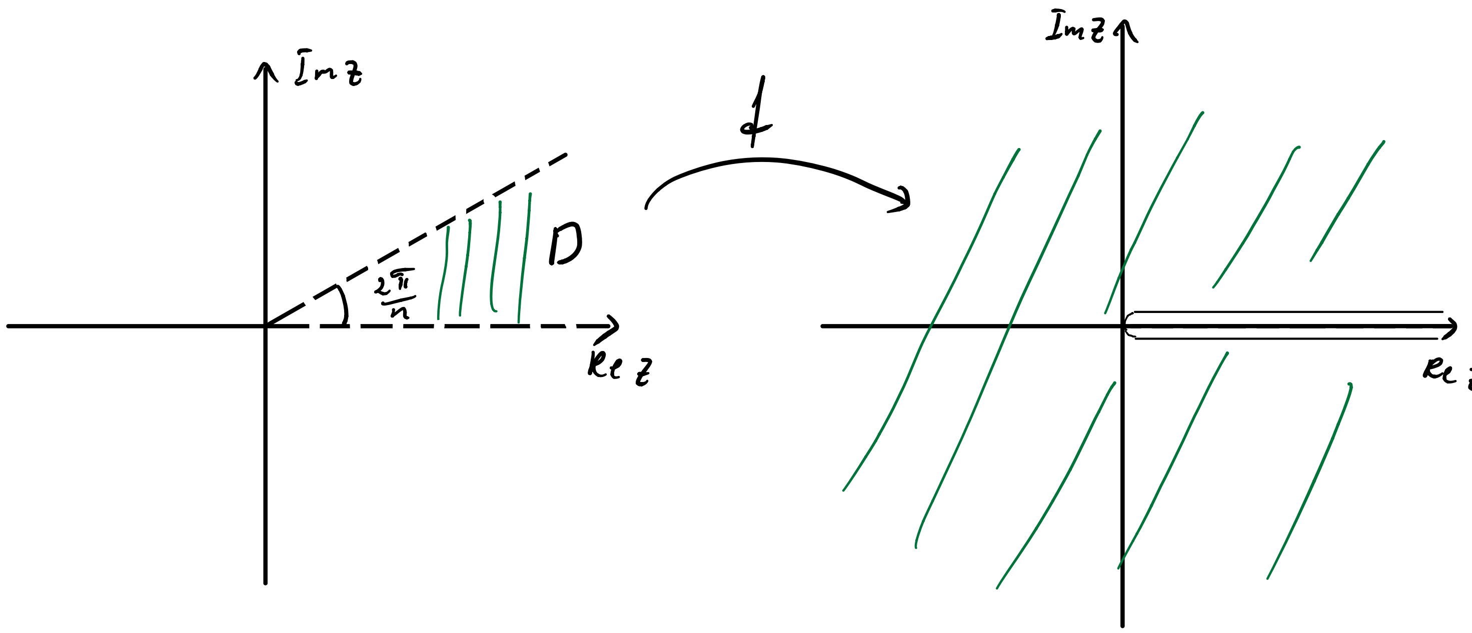
$$z = r(\cos\varphi + i \sin\varphi),$$

then
$$z^n = r^n(\cos n\varphi + i \sin n\varphi)$$
by de Moivre's formula.



Hence, if $z_1, z_2 \in \mathbb{C}$, such that
$$|z_1| = |z_2|$$
and
$$\arg z_1 = \arg z_2 + k\frac{2\pi}{n},$$
then $z_1^n = z_2^n$.

This implies that f is not bijective on $\mathbb{C}$.



The function f is bijective from
$$D = \{z : 0 < \arg z < \tfrac{2\pi}{n}\} \text{ to } \mathbb{C} \setminus \{z = x+iy : x \geq 0\}.$$

6) Exponential function

We define the function

$$e^z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n .$$

Let us show that there exists this limit for any $z \in \mathbb{C}$. Let $z = x+iy$. We observe that

$$\left|\left(1+\frac{z}{n}\right)^n\right| = \left|\left(1+\frac{x}{n}+i\frac{y}{n}\right)^n\right| =$$

$$= \left(\left(1+\frac{x}{n}\right)^2+\frac{y^2}{n^2}\right)^{\frac{n}{2}} = \left(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)^{\frac{n}{2}}$$

and

$$\arg\left(1+\frac{z}{n}\right)^n = n\arctan\frac{\frac{y}{n}}{1+\frac{x}{n}} .$$

Hence, $$\lim_{n\to\infty}\left|\left(1+\frac{z}{n}\right)^n\right| = \lim_{n\to\infty}\left(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)^{\frac{n}{2}} =$$

$$= \lim_{n\to\infty}\underbrace{\left(1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)^{\frac{1}{\frac{2x}{n}+\frac{x^2+y^2}{n^2}}}}_{\to e}{}^{\cdot\left(\frac{2x}{n}+\frac{x^2+y^2}{n^2}\right)\cdot\frac{n}{2}} =$$

$$= e^x$$

$$\lim \arg\left(1+\frac{z}{n}\right)^n = \lim_{n\to\infty} n\arctan\frac{\frac{y}{n}}{1+\frac{x}{n}} =$$

$$= \lim_{n\to\infty}\underbrace{\frac{\arctan \frac{y}{n}/1+\frac{x}{n}}{\frac{y}{n}/1+\frac{x}{n}}}_{\to 1}\cdot\left(\frac{y}{n}/1+\frac{x}{n}\right)\cdot n = y$$

Therefore $$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

In particular, $e^{iy} = \cos y + i \sin y$ (Euler formula)

Exercise 15.1 Show that the function

$$f(z) = e^z, \quad z \in \mathbb{C}$$

is holomorphic in $\mathbb{C}$ and

$$f'(z) = e^z.$$

c) The trigonometric functions

The Euler formula $e^{iy} = \cos y + i \sin y$,

$$e^{-iy} = \cos y - i \sin y \quad \forall y \in \mathbb{R}$$

gives $\cos y = \dfrac{e^{iy} + e^{-iy}}{2}$, $\sin y = \dfrac{e^{iy} - e^{-iy}}{2i}$.

we define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

$$z \in \mathbb{C}.$$

Exercise 15.2 Show that

a) $\sin^2 z + \cos^2 z = 1$, $\cos z = \sin\left(z + \frac{\pi}{2}\right)$.

b) $f(z) = \cos z$ and $g(z) = \sin z$ are holomorphic functions in $\mathbb{C}$ and

$$(\cos z)' = -\sin z, \quad (\sin z)' = \cos z.$$

The trigonometri functions of a complex variable are closely related to the hyperbolic ones defined by

$$\cosh z = \frac{e^{z}+e^{-z}}{2}, \quad \sinh z = \frac{e^{z}-e^{-z}}{2}.$$

So,

$$\cosh z = \cos iz, \quad \sinh z = -\sin iz$$
$$\cos z = \cosh iz, \quad \sin z = -i \sinh iz$$

Exercise 15.3 Show that

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$$