Complex Analysis

14. Holomorphic functions

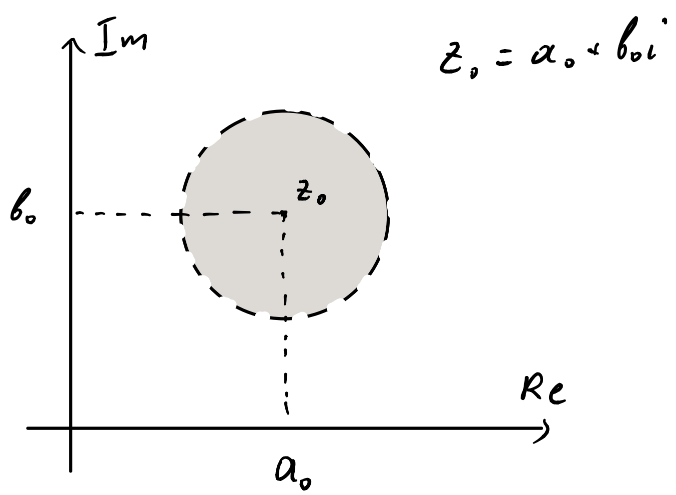
1. Basic notions

Let $z = x + yi \in \mathbb{C}$ be a complex number. Here $i^2 = -1$. We denote

- $\operatorname{Re} z := x$ – real part of z
- $\operatorname{Im} z := y$ – imaginary part of z
- $\bar{z} = x - yi$ – conjugate of z
- $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2}$ – absolute value of z.

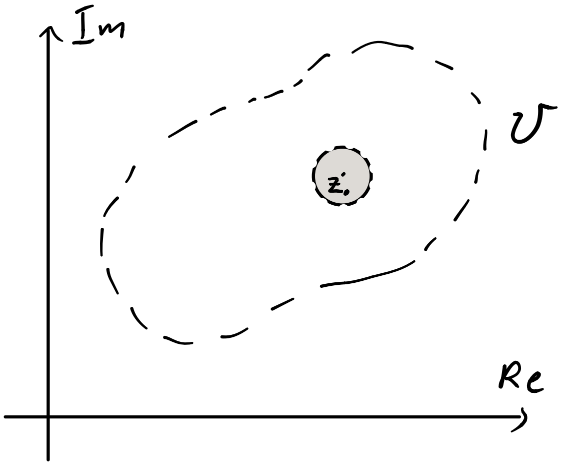
- We will denote

$$B_r(z_0) = \{ z \in \mathbb{C} : |z - z_0| < r \} =$$
$$= \{ z = a + bi : (x - x_0)^2 + (y - y_0)^2 < r \}$$



an open ball in $\mathbb{C}$ with center z_0 and radius r.

- A set $U \subseteq \mathbb{C}$ is open in $\mathbb{C}$ if each point $z_0 \in U$ belongs



to U together with some $B_r(z_0)$ i.e. $\forall z_0 \in U \ \exists r > 0$ s.t. $B_r(z_0) \subseteq U$.

2. Differentiable functions

We will consider functions from $\mathbb{C}$ to $\mathbb{C}$. So, let U be an open subset of $\mathbb{C}$ and $f: U \to \mathbb{C}$ be a complex function.

Def 14.1 a) Let $z_0 \in U$. if there exists a limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0),$$

i.e. $\forall \varepsilon > 0 \ \exists \delta > 0 : \forall z \in B_\delta(z_0),\ z \neq z_0$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon,$$

then f is called *complex differentiable at z_0* and $f'(z_0)$ is called *the derivative of f at z_0*

b) If f is complex differentiable for every $z_0 \in U$, we say that f is holomorphic in U

c) We say that f is holomorphic at z_0 if f is complex differentiable in a neighbourhood of z_0 (in some open set U_0 containing z_0)

Ex 14.1 $f(z) = z^2 = (x+yi)^2 = x^2 - y^2 + 2xyi$
$z = x+yi \in \mathbb{C}$

The function f is differentiable on $\mathbb{C}$.
Indeed,

$$\lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{(z-z_0)(z+z_0)}{z - z_0} =$$

$$= \lim_{z \to z_0} (z + z_0) = 2z_0.$$

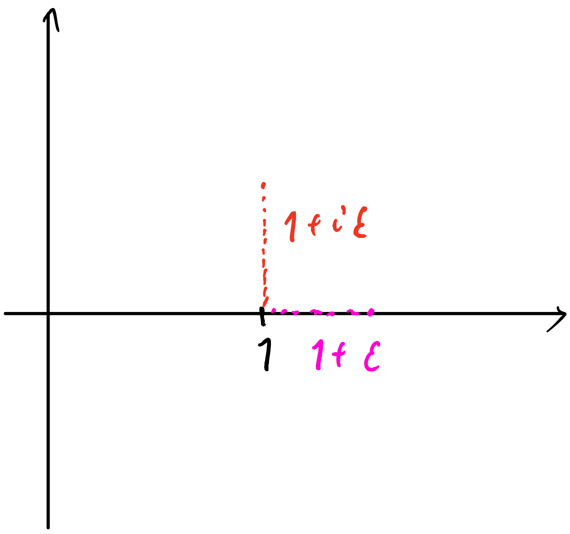
So, $f'(z) = 2z$.

Ex 14.2 The function

$$f(z) = |z|^2 = x^2 + y^2, \quad z = x+yi \in \mathbb{C}$$

is not differentiable at $z_0 = 1$.

In order to show this, we consider



$$\lim_{\varepsilon\to 0} \frac{|1+\varepsilon|^2 - 1}{\varepsilon} =$$

$$= \lim_{\varepsilon\to 0} \frac{2\varepsilon + \varepsilon^2}{\varepsilon} = 2$$

$$\lim_{\varepsilon\to 0} \frac{|1+i\varepsilon|^2 - 1}{i\varepsilon} = \lim_{\varepsilon\to 0} \frac{1+\varepsilon^2-1}{i\varepsilon} = 0$$

This shows that $f'(1)$ does not exists

Proposition 14.1 If f is differentiable at z_0 then f is continuous at z_0

Proof We set

$$r(z_0, z) := \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \xrightarrow[z\to z_0]{} 0$$

Then

$$f(z) = f(z_0) + f'(z_0)\underbrace{(z-z_0)}_{\substack{\downarrow z\to z_0 \\ 0}} + \underbrace{r(z_0,z)}_{\substack{\downarrow z\to z_0 \\ 0}}(z-z_0)$$

then $f(z) \to f(z_0)$.

Propozition 14.2 a) If f and g are holomorphic in U, then $f \pm g$, $f \cdot g$, $\frac{f}{g}$ (if $g \neq 0$) are holomorphic on U and

- $(f \pm g)' = f' \pm g'$

- $(fg)' = f'g + fg'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

6) (chain rule) if $f: U \to V$, $g: V \to \mathbb{C}$, where U, V are open sets, are holomorphic, then $g \circ f$ is holomorphic and

$$(g \circ f)'(z) = g'(f(z))\, f'(z).$$

Proof Are the same as in real case.

3. Cauchy - Riemann equations

Let us identify the complex field $\mathbb{C}$ and $\mathbb{R}^2$ with $z = x + iy$, that is every complex number z corresponds to an ordered pair (x, y)

$$\mathbb{C} \ni z = x + iy \longleftrightarrow (x, y) \in \mathbb{R}^2$$

Then a complex function

$$w = f(z)$$

corresponds to a functions

$$u = u(x, y) = \operatorname{Re} f(z)$$
$$v = v(x, y) = \operatorname{Im} f(z),$$

that is
$$f(z) = u(x,y) + i v(x,y)$$

Theorem 14.1 (Cauchy - Riemann)
Let $f = u + iv : U \to \mathbb{C}$ be a function, $U \subseteq \mathbb{C}$ be open and $z_0 = x_0 + iy_0 \in U$.
The following statements are equivalent:

a) f is complex differentiable at z_0;

b) u, v are real differentiable at (x_0, y_0) and Cauchy - Riemann equation are satisfied

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x}(x_0,y_0) &= \frac{\partial v}{\partial y}(x_0,y_0) \\ \frac{\partial u}{\partial y}(x_0,y_0) &= -\frac{\partial v}{\partial x}(x_0,y_0) \end{aligned}}$$

In this case
$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$$
$$= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0).$$

Proof. $\Rightarrow$) Let f be differentiable at z_0 and
$$f'(z_0) = a + ib$$

We set $\Delta f(z_0) = f(z_0 + \Delta z) - f(z_0)$

$\Delta z = z - z_0$

Then $\displaystyle\lim_{\Delta z \to 0} \frac{\Delta f(z_0)}{\Delta z} = f'(z_0) = a + ib.$

We set $\displaystyle r(z_0, \Delta z) := \frac{\Delta f(z_0)}{\Delta z} - f'(z_0) \underset{\Delta z \to 0}{\longrightarrow} 0$

Let us rewrite

$$\Delta f(z_0) = f'(z_0)\Delta z + r(z_0, \Delta z)\cdot \Delta z \qquad (14.1)$$

Then $\Delta f(z_0) = f(z_0 + \overset{\Delta x + i\Delta y}{\overline{\Delta z}}) - f(\overset{= x_0 + i y_0}{z_0}) =$

$= \big(u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)\big) +$

$+ i\big(v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)\big) =$

$=: \Delta u(x_0, y_0) + i\Delta v(x_0, y_0)$

We also set $r = r_1 + i r_2$. Hence, by (1),

$\Delta u(x_0, y_0) + i\Delta v(x_0, y_0) = (a + ib)(\Delta x + i\Delta y) +$

$+ (r_1 + i r_2)(\Delta x + i\Delta y).$

Consequently

$$\Delta u(x_0, y_0) = a\Delta x - b\Delta y + \varepsilon_1 \Delta x - \varepsilon_2 \Delta y$$

$$\Delta v(x_0, y_0) = b\Delta x + a\Delta y + \varepsilon_2 \Delta x + \varepsilon_2 \Delta y$$

$$\Rightarrow \quad a = \frac{\partial u}{\partial x}(x_0, y_0)$$

$$-b = \frac{\partial u}{\partial y}(x_0, y_0)$$

$$b = \frac{\partial v}{\partial x}(x_0, y_0)$$

$$a = \frac{\partial v}{\partial y}(x_0, y_0)$$

by the definition of derivatives u, v.

$\Leftarrow$) Let u, v satisfy Cauchy-Riemann equations. We prove that f is differentiable at z_0. Since u is differentiable at (x_0, y_0)

$$\Delta u(x_0, y_0) = a\Delta x + b\Delta y + \alpha\sqrt{\Delta x^2 + \Delta y^2},$$

where $\alpha \to 0$, $(\Delta x, \Delta y) \to (0, 0)$

$$\Delta v(x_0, y_0) = -b\Delta x + a\Delta y + \beta\sqrt{\Delta x^2 + \Delta y^2},$$

where $\beta \to 0$, $(\Delta x, \Delta y) \to (0, 0)$.

Here $a = \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$,

$b = \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$

Then

$$\Delta f(z_0) = \Delta u(x_0, y_0) + i\,\Delta v(x_0, y_0) =$$

$$= (a - ib)\Delta x + (\overbrace{b + ia}^{=(a-ib)i})\Delta y + \underbrace{(\alpha + i\beta)\sqrt{\Delta x^2 + \Delta y^2}}_{\overset{\shortparallel}{r\,\Delta z}}$$

Hence, $\Delta f(z_0) = (a - ib)\Delta z + r\Delta z.$

We need to check that $r \to 0$, $\Delta z \to 0$.

So,
$$0 \le |r| = \left|\frac{(\alpha + i\beta)|\Delta z|}{\Delta z}\right| \le \big(|\alpha| + |\beta|\big)\frac{|\Delta z|}{|\Delta z|} =$$

$$= |\alpha| + |\beta| \to 0, \quad \Delta z \to 0.$$

Hence, f is differentiable at z_0 and

$$f'(z_0) = a - ib.$$

$\blacksquare$