# 12. Gauss-Ostrogradskii theorem

Let $S$ be a piecewise-smooth surface surrounding a compact domain $V$ in $\mathbb{R}^3$ and oriented by the outgoing normal (such orientation will be called positive).
Let $\vec{F} = (P, Q, R)$ be smooth vector field in the closed domain $\bar{V}$.

We recall

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is the divergence of $\vec{F}$.
We also recall from the previous lecture that the surface integral of the vector field $\vec{F}$ is defined as

$$\iint_S \vec{F} \cdot ds = \iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

$$= \iint_D \left( P \frac{\partial(y,z)}{\partial(u,v)} + Q \frac{\partial(z,x)}{\partial(u,v)} + R \frac{\partial(x,y)}{\partial(u,v)} \right) du\,dv$$

where $r(u,v) = (x(u,v), y(u,v), z(u,v))$ is a parametrization of $S$, and

$S$ is oriented in the direction $r_u \times r_v$.

Theorem 12.1 (Gauss – Ostrogradskii)
Let $\vec{F}$, $S$, $V$ be as before. Then

$$(12.1) \quad \boxed{\iint_S \vec{F} \cdot ds = \iiint_V \operatorname{div} F \, dx\,dy\,dz}$$

Remark 12.1 Let $B_\varepsilon(p)$ and $S_\varepsilon(p)$ denote a ball and sphere with center $p \in \mathbb{R}^3$ and radius $\varepsilon$. The Gauss-Ostrogradskii theorem implies

$$\iiint_{B_\varepsilon(p)} \operatorname{div} F \, dx\,dy\,dz = \iint_{S_\varepsilon(p)} \vec{F} \cdot ds$$

flux of $\vec{F}$ across $S_\varepsilon(p)$ in the outward direction

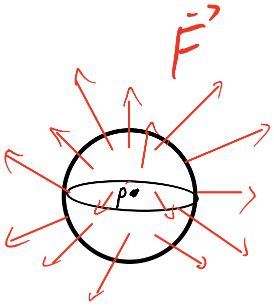
Using the mean-value theorem for domain integrals, we have

$$\operatorname{div} F(\tilde{p}) \operatorname{Vol}(B_\varepsilon(p)) = \iint_{S_\varepsilon(p)} \vec{F} \cdot ds,$$

where $\tilde{p}$ is a point from $B_\varepsilon(p)$.
Then, by continuity of $\operatorname{div} F$, we

have

$$(12.2)\quad \operatorname{div} F(p) = \lim_{\varepsilon \to 0} \frac{1}{\operatorname{vol}(B_\varepsilon(p))} \iint_{S_\varepsilon(p)} \vec{F} \cdot d\vec{S}$$



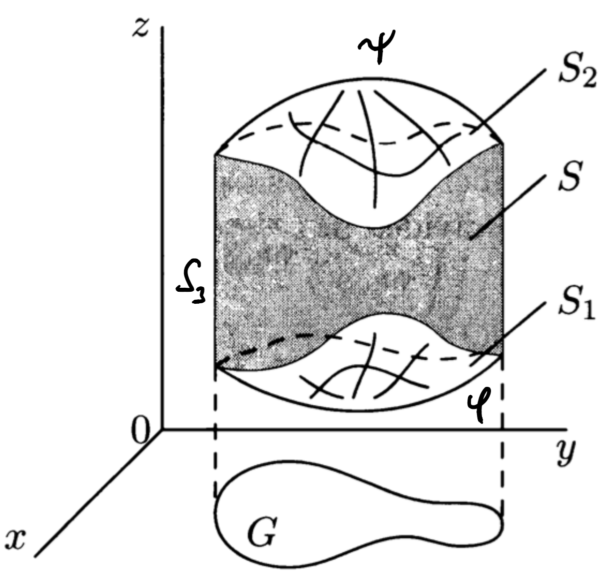

In particular, $\operatorname{div} \vec{F}$ is independent of coordinate system (although it is defined as sum of partial derivatives with respect to a fixed Cartesian coordinate system).

Remark 12.2 The fraction in the right hand side of (12.2) can be interpreted as the mean intensity (per unit volume) of sources in the ball $B_\varepsilon(p)$, that is, $\operatorname{div} \vec{F}(p)$, is the specific intensity (per unit volume) of the source or sink at the point $p$.

Remark 12.3 If $\operatorname{div} F(p)$ is positive for $p \in \mathbb{R}^3$, then $p$ is a source, if negative, then a sink. The Gauss-Ostrogradskii theorem states that the flux of $\vec{F}$ across

S equals to the "sum" of all flows from sources in V minus the "sum" of all flows to sinks in V.

Proof of Th 12.1 By the linearity of the right and left hand sides of (12.1) with respect to the vector field $\vec{F}$, it suffices to prove the theorem for vector fields $(P,0,0)$, $(0,Q,0)$, $(0,0,R)$ and then add up the obtained results. We only consider $F=(0,0,R)$.




(12.1) can be derived by repeating the derivation of Green's formula. We express V as a union of pairwise almost-disjoint sets $V_1, \dots, V_k$ of the form $V=\{(x,y,z):\ (x,y)\in G,\ \varphi(x,y)\le z\le \psi(x,y)\}$.

By the fundamental theorem of calculus

$$\iiint\limits_V \frac{\partial R}{\partial z}\,dx\,dy\,dz=\iint\limits_G dx\,dy\int_{\varphi(x,y)}^{\psi(x,y)}\frac{\partial R}{\partial z}\,dz=$$

$$= \iint_G R(x,y,\psi(x,y))\,dx\,dy - \iint_G R(x,y,\varphi(x,y))\,dx\,dy.$$

Let $S_1 = \{(x,y,z) : (x,y) \in G,\ z = \varphi(x,y)\}$

and $S_2 = \{(x,y,z) : (x,y) \in G,\ z = \psi(x,y)\}$.

We note that on both surfaces the z-coordinate of the vector $r_x \times r_y$ equals 1. Indeed,

$$r_x = (1, 0, \varphi_x), \quad r_y = (0, 1, \varphi_y), \qquad z\text{-coordinate}: \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Thus, $r_x \times r_y$ is oriented outwards on $S_2$ and inwards on $S_1$. We can continue

$$\iint_G R(x,y,\psi(x,y))\,dx\,dy - \iint_G R(x,y,\varphi(x,y))\,dx\,dy$$

$$= \iint_{S_2} \vec{F} \cdot ds + \iint_{S_1} \vec{F} \cdot ds.$$

Since, the normal to the side boundary of V is orthogonal to $F = (0, 0, R)$, The flux of $\vec{F}$ through the side boundary $S_3$ equals 0.

So, $$\iiint_V \frac{\partial R}{\partial z}\,dx\,dy\,dz = \iint_{S_1} \vec{F} \cdot ds + \iint_{S_2} \vec{F} \cdot ds + \underbrace{\iint_{S_3} \vec{F} \cdot ds}_{=0}$$

$$= \iint_S \vec{F} \cdot dS = \iint_S R\, dx\, dy.$$

**Corollary 12.1** If $V$ is a connected set with boundary consisting of piecewise-smooth surfaces $S, S_1, \dots S_k$ (here $S$ is the outer boundary and $S_i$, $i=1,\dots,k$ are boundaries of holes in $V$) all oriented by outgoing normals, then

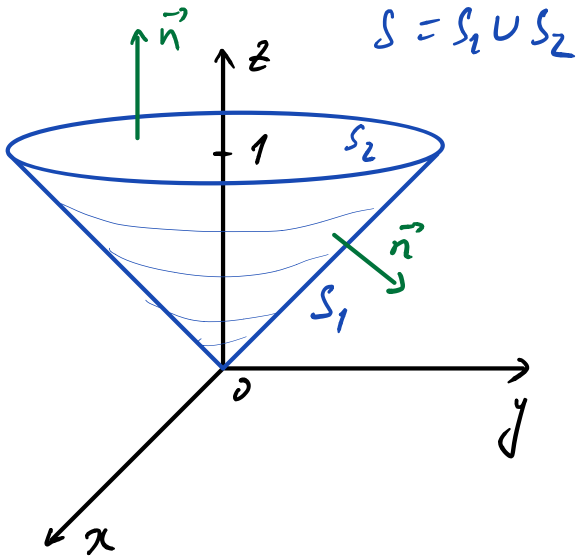
$$\iiint_V \operatorname{div} F\, dx\, dy\, dz = \iint_S F \cdot dS + \sum_{i=1}^{k} \iint_{S_k} F \cdot dS$$

**Example 12.1** Let $S_1$ be the lateral surface of the cone $x^2+y^2 \le z^2 \le 1$ oriented outwards. Let $\vec{F} = (x^3, y^3, z^3)$. In order to compute

$$\iint_{S_1} \vec{F} \cdot dS$$

we use the Gauss-Ostrogradskii theorem. Let $V = \{(x,y,z) :\ x^2+y^2 \le z^2 \le 1\}$.

Then $\displaystyle\iint_S F \cdot dS = \iiint_V \operatorname{div} \vec{F}\, dx\, dy\, dz \quad =$




$$= 3 \iiint_V (x^2 + y^2 + z^2)\,dx\,dy\,dz \;\circledcirc$$

We apply the change of coordinates

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases} \qquad y = \rho$$

$0 \le z \le 1, \quad 0 \le \varphi \le 2\pi,$

$0 \le \rho \le z$

So, $\circledcirc \; 3\int_0^1 dz \int_0^{2\pi} d\varphi \int_0^z (\rho^2 + z^2)\rho\,d\rho =$

$$= 6\pi \int_0^1 \left(\frac{\rho^4}{4} + \frac{z^2\rho^2}{2}\right)\Bigg|_0^z dz = 6\pi \int_0^1 \left(\frac{z^4}{4} + \frac{z^4}{2}\right) dz$$

$$= \frac{18}{4}\pi \frac{z^5}{5}\Bigg|_0^1 = \frac{18}{20}\pi = \frac{9}{10}\pi.$$

Hence

$$\iint_{S_1} \vec{F} \cdot d\vec{s} = \frac{9}{10}\pi - \iint_{S_2} \vec{F} \cdot d\vec{s}.$$

The parametrization of $S_2$ is

$$\begin{cases} x = u \\ y = v \\ z = 1 \end{cases} \qquad (u,v) \in D = \{(x,y) : x^2 + y^2 \le 1\}$$

Then, the normal vector $n = (0, 0, 1)$

Therefore,

$$\iint_{S_2} \vec{F} \cdot dS = \iint_{S_2} \underset{\substack{\parallel \\ 1 \text{ on } S_2}}{z^3}\, dS = \iint_{S_2} dS =$$

$$= \text{Area}(S_2) = \pi.$$

So, we have obtained

$$\iint_{S_1} F \cdot dS = \frac{9\pi}{10} - \pi = -\frac{\pi}{10}.$$