1

10. Surface integral of a scalar field.

1. Surfaces.

We start from the definition of a surface.

Def 10.1 A surface in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ that be parametrized by a continuous vector function r:

$$S = \{ r(u,v) = (x(u,v), y(u,v), z(u,v)), \; (u,v) \in \overline{D} \},$$

where D is a bounded domain of $\mathbb{R}^2$ and $r(u,v) \neq r(u',v')$ for all $(u,v) \neq (u',v')$ in D. (We allow that the injectivity property of r may fail on the boundary of D)

Ex 10.1 a) A parametrization for a sphere $x^2+y^2+z^2=a$ is

$$\begin{cases} x = a\cos\varphi\cos\psi \\ y = a\sin\varphi\cos\psi & \varphi \in [0,2\pi], \\ z = a\sin\psi & \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$$

$$D = (0,2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$$

(2) b) We will also consider surfaces that are graphs of continuous functions. Let f be a continuous function on D

$$S = \{ (x, y, f(x,y)), (x,y) \in \bar{D} \}.$$

<u>Def 10.2</u> If a surface is parametrized by a continuously differentiable vector function, then it is called a <u>continuously differentiable surface</u>.

Here we will consider continuously diff. surfaces.

<u>Def 10.3</u> For a surface

$$S = \{ r(u,v) = (x(u,v), y(u,v), z(u,v)), (u,v) \in \bar{D} \}$$

and a point $(u_0, v_0) \in D$, the lines

$$\{ r(u, v_0), (u, v_0) \in \bar{D} \} \text{ and } \{ r(v_0, v), (u_0, v) \in D \}$$

are called (u- and v-) curvilinear coordinates on S at $r(v_0, v_0)$.

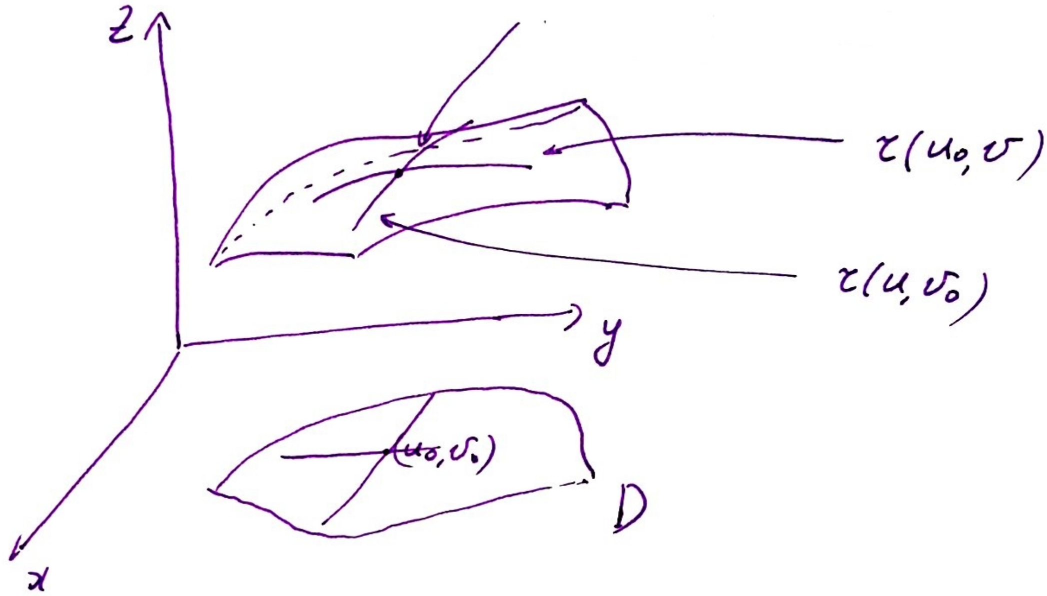
A tangent vectors to those lines are denoted by

③

$$r_u = r_u(u_0, v_0) = \frac{\partial r}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

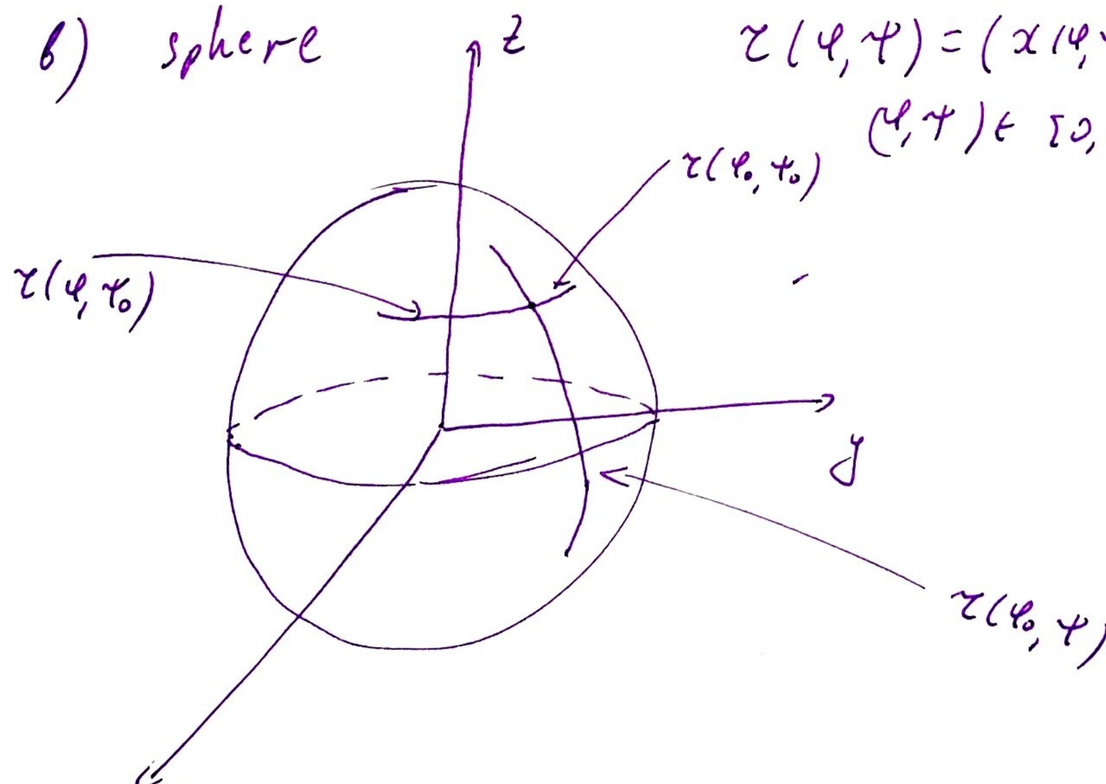
$$r_v = r_v(u_0, v_0) = \frac{\partial r}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

<u>Ex 10.2</u> a) Graph of a continuous function
$r(u,v) = (u, v, f(u,v))$, $(u,v) \in \overline{D}$



b) sphere

$r(\varphi, \psi) = (x(\varphi,\psi), y(\varphi,\psi), z(\varphi,\psi))$
$(\varphi, \psi) \in [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$



(4)

Here we will consider only surfaces such that r_u, r_v are non-colinear, that is,

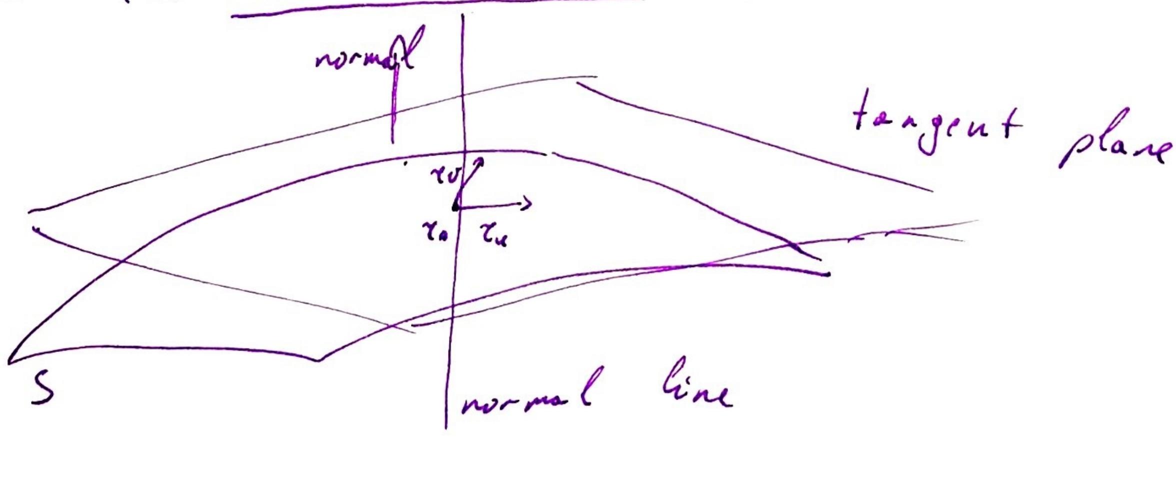
$$r_u \times r_v \neq 0.$$

In this case r_u, r_v span a plain in $\mathbb{R}^3$, called the <u>tangent plane</u> to S at $r(u_0, v_0)$

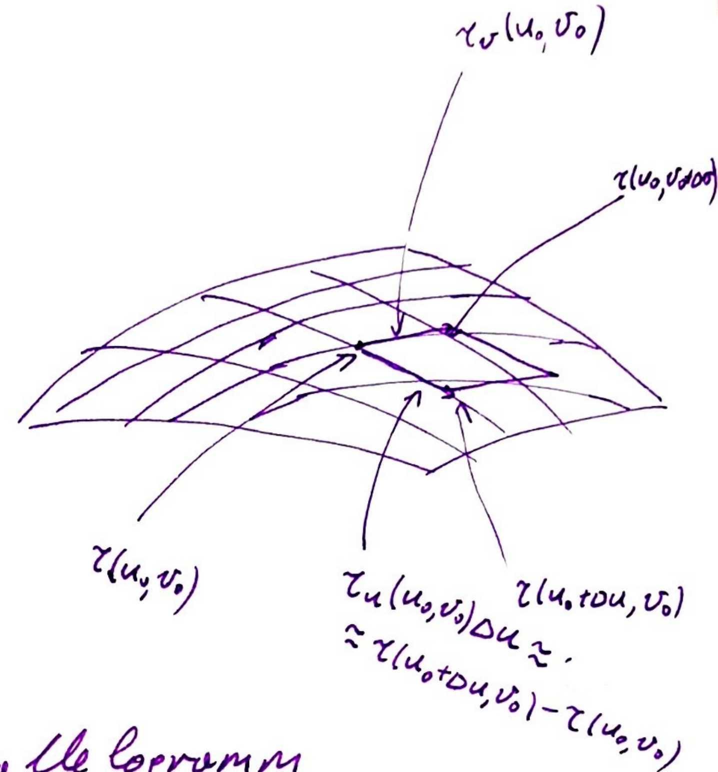
<u>Remark 10.1</u> The equation for the tangent plane to S at $r(u_0, v_0) = (x_0, y_0, z_0)$ is

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_u(u_0,v_0) & y_u(u_0,v_0) & z_u(u_0,v_0) \\ x_v(u_0,v_0) & y_v(u_0,v_0) & z_v(u_0,v_0) \end{vmatrix} = 0.$$

<u>Def 10.4</u> The line orthogonal to the tangent plane at $r_0 \in S$ is called the normal line to S at r_0. Every non-zero vector parallel to the normal line at r_0 is called the <u>normal vector</u> to S at r_0



2. Surface area.



Let Γ_i be a rectangle $[u_i, u_i+\Delta u_i]\times[v_i, v_i+\Delta v_i]$ in D and S_i its image in S.

The area of S_i can be well approximated by the area of the parallelogramm in $\mathbb{R}^3$ spanned by the vectors $r_u(u_i, v_i)\Delta u_i$, $r_v(u_i, v_i)\Delta v_i$, as $\Delta u, \Delta v \to 0$.

The area of the rectangle equals

$$\|r_u \times r_v\| \cdot \Delta u\, \Delta v.$$

Using the Riemann sum approximations, we have

$$\text{Area}(S) = \iint_D \|r_u \times r_v\|\, du\, dv.$$

Note, that

angle between r_u, r_v

$$\|r_u \times r_v\|^2 = \|r_u\|^2\|r_v\|^2 \sin^2\theta = \|r_u\|^2\|r_v\|^2 - \|r_u\|^2\|r_v\|^2\cos^2\theta = \|r_u\|^2\|r_v\|^2 - (r_u, r_v)^2 = EG - F^2,$$

where
$$E := \|r_u\|^2, \quad F := (r_u, r_v), \quad G := \|r_v\|^2.$$

So,
$$\text{Area}(S) = \iint_D \sqrt{EG - F^2}\, du\, dv.$$

<u>Remark 10.2</u> The area does not depend on the parametrization (Consider as an exercise).

<u>Example 10.3</u> a) Sphere:
$$\begin{cases} x = a\cos\varphi\cos\psi \\ y = a\sin\varphi\cos\psi \\ z = a\sin\psi \end{cases}, \quad \begin{array}{l} \varphi \in [0, 2\pi] \\ \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{array}$$

$$r_\varphi = (-a\sin\varphi\cos\psi, \; a\cos\varphi\cos\psi, \; 0)$$

$$E = \|r_\varphi\|^2 = a^2\sin^2\varphi\cos^2\psi + a^2\cos^2\varphi\cos^2\psi = $$
$$= a^2\cos^2\psi$$

Similarly

$F = 0$, $G = a^2$. So $\sqrt{EG - F^2} = a^2\cos\psi$

b) Graph of a function
$$\begin{cases} x = x \\ y = y \\ z = f(x,y) \end{cases} \qquad \begin{array}{l} r_x = (1, 0, f_x) \\ r_y = (0, 1, f_y) \end{array}$$

Hence

$$E = 1^2 + f_x^2 = 1 + f_x^2$$

$$F = f_x f_y$$

$$G = 1^2 + f_y^2 = 1 + f_y^2$$

Hence, $\sqrt{EG - F^2} = \sqrt{(1+f_x^2)(1+f_y^2) - f_x^2 f_y^2} =$

$$= \sqrt{1 + f_x^2 + f_y^2}$$

3. Surface integral of a scalar field

Let $S = \{ r(u,v), (u,v) \in \bar{D} \}$ be a continuously differentiable surface in $\mathbb{R}^3$ and f be a real-valued function defined on S.

Def 10.5 The integral of f over S is denoted by and defined as

$$\iint_S f \, dS = \iint_D f(x(u,v), y(u,v), z(u,v)) \cdot \sqrt{EG - F^2} \, du \, dv.$$

Remark 10.3 A physical interpretation of the integral of f over S for non-negative f is the mass of surface S with density f.

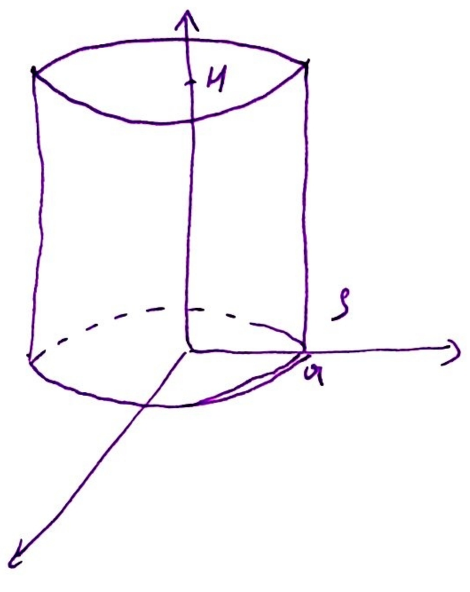
Lemma 10.1 The definition of $\iint_S f \, dS$ is independent of parametrization S.

Ex. 10.4 We compute

$$I = \iint_S \frac{dS}{\sqrt{x^2+y^2+z^2}}, \text{ where } S \text{ is}$$

the leteral surdace od the cylinder

$$\begin{cases} x = a\cos u \\ y = a\sin u & 0 \le u \le 2\pi \\ z = v & 0 \le v \le H \end{cases}$$



We first compute

$\sqrt{EG-F^2} = a$

$r_u = (-a\sin u, a\cos u, 0)$

$r_v = (0, 0, 1)$

$E = a^2$

$G = 1$

$F = 0$

Thus,

$$I = \int_0^{2\pi}\int_0^H \frac{1}{\sqrt{a^2+v^2}}\, a\, du\, dv = 2\pi a \ln\left(v+\sqrt{a^2+v^2}\right)\Big|_0^H$$

$$= 2\pi a \ln \frac{H+\sqrt{a^2+H^2}}{a}$$