7] Line integral of scalar and vector field

1. Line integral of a scalar field

Recall the definition given in Lecture 6:

Let γ be a rectifiable curve with length L.
Let $x(s)$, $0 \le s \le L$, be the natural parametrization of γ. $\Gamma = \{ x(s) : 0 \le s \le L \}$.
Let $f : \Gamma \to \mathbb{R}$.

Def of line integral

$P = \{ s_0, \dots, s_n \}$ partition of $[0, L]$

$$\int_\gamma f \, ds := \lim_{\lambda(P) \to 0} \sum_{i=1}^{n} f(x(s_i))(s_i - s_{i-1})$$

if the limit exists.

$$= \int_0^L f(x(s)) \, ds \quad (\text{Riemann integral}).$$

Some properties of the line integral

1) $\int_\gamma ds = L$, where L is the length of γ
(it follows from def)

2) If f is bounded and continuous, then $\int f \, ds$ exists
(it follows from Lebesgue criterion).

3) Let $\tilde{\gamma}(t)$, $a \le t \le b$, be another parametrization of γ, that is regular: $\tilde{\gamma}'(t) \neq 0$.

Then $$\boxed{\int_\gamma f\,ds = \int_a^b f(\tilde{\gamma}(t))\,\|\tilde{\gamma}'(t)\|\,dt}.$$

Proof:

Let $s(t) = \int_a^t \|\tilde{\gamma}'(\tau)\|\,d\tau$, $a \le t \le b$.

Then $\tilde{\gamma}(t) = x(s(t))$ for all $t \in [a,b]$.

Change of variable $s = s(t)$:

$$\int_\gamma f\,ds = \int_0^L f(x(\underbrace{s}_{s(t)}))\,\underbrace{ds}_{\|\tilde{\gamma}'(t)\|\,dt} = \int_a^b f(\tilde{\gamma}(t))\,\|\tilde{\gamma}'(t)\|\,dt.$$

$\square$

4) If γ^R is the time reversal of γ:

$y(s)$, $0 \le s \le L$, natural par. of γ^R is given by

$y(s) = x(L-s)$.

Then $$\boxed{\int_{\gamma^R} f\,ds = \int_\gamma f\,ds}.$$

Proof:

$$\int_{\gamma^R} f\,ds = \int_0^L f(y(s))\,ds = \int_0^L f(x(L-s))\,ds = \int_0^L f(x(s))\,ds$$
$$= \int_\gamma f\,ds.$$

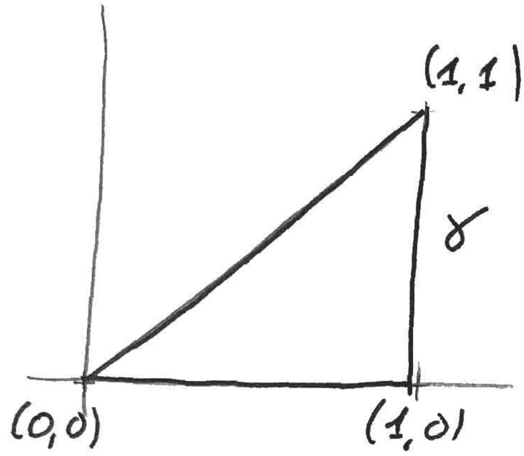
$\square$

5) $a, b \in \mathbb{R}$ $\quad f, g : \Gamma \to \mathbb{R}$

$$\int_\gamma (af + bg)\,ds = a\int_\gamma f\,ds + b\int_\gamma g\,ds.$$

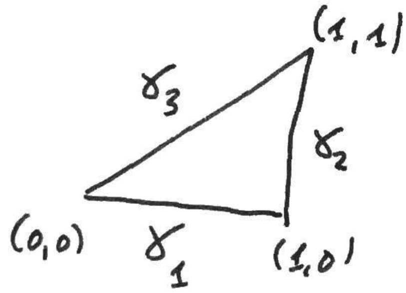
6) $\left|\int_\gamma f\, ds\right| \leq M\,\ell(\gamma)$ where $M = \sup_I |f|$.

Example :



γ boundary of this triangle in $\mathbb{R}^2$

$$I = \int_\gamma (x+y)\, ds \quad ?$$



γ_1 : segment connecting (0,0) and (1,0)

γ_2 : ——————— (1,0) and (1,1)

γ_3 : ——————— (1,1) and (0,0)

· Parametrization of γ_1: $\{(t,0),\ 0 \leq t \leq 1\}$ (natural par.)

$$\int_{\gamma_1} (x+y)\, ds = \int_0^1 t \quad dt = \frac{1}{2}.$$

· Parametrization of γ_2 : $\{(1,t),\ 0 \leq t \leq 1\}$ (natural par.)

$$\int_{\gamma_2} (x+y)\, ds = \int_0^1 (1+t)\, dt = \left[t + \frac{t^2}{2}\right]_0^1 = \frac{3}{2}.$$

· Param. of γ_3 : $\{(t,t),\ 0 \leq t \leq 1\}$ (not natural)

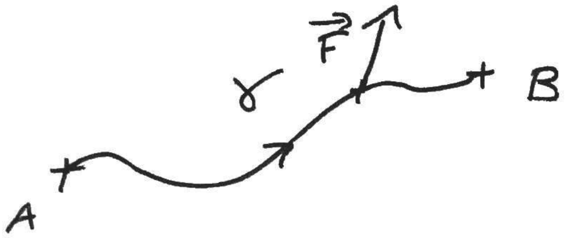
$$\int_{\gamma_3} (x+y)\, ds = \int_0^1 (t+t)\sqrt{1^2+1^2}\, dt = \sqrt{2}\left[2\frac{t^2}{2}\right]_0^1 = \sqrt{2}.$$

(the orientations of $\gamma_1, \gamma_2, \gamma_3$ have no importance by property 4)

Finally $I = I_1 + I_2 + I_3 = 2 + \sqrt{2}$.

2. Line integral of a vector field.

In dimension 2 :



$\vec{F} = (P, Q)$ vector field on $\mathbb{R}^2$

$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \mapsto (P(x,y), Q(x,y))$.

γ rectifiable curve on $\mathbb{R}^2$, with parametrization

$\gamma(t) = (x(t), y(t))$, $a \leq t \leq b$.

Assume that γ is regular ($\|\gamma'(t)\| > 0$ for all t).

Definition : The line integral of $\vec{F}$ along γ is denoted by $\int_\gamma \vec{F} \cdot ds$ and defined as

$$\int_\gamma \vec{F} \cdot ds = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_a^b \Big(P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t) \Big)\, dt.$$

usual Riemann integrals.

In dimension 3:

$$\vec{F} = (P, Q, R)$$

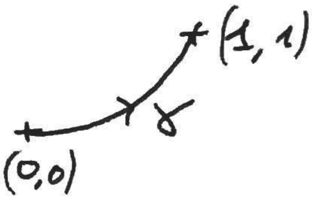
$\gamma(t) = (x(t), y(t), z(t))$, $a \le t \le b$ curve on $\mathbb{R}^3$

$$\int_\gamma \vec{F} \cdot ds = \int_a^b \Big(P(\gamma(t))\, x'(t) + Q(\gamma(t))\, y'(t) + R(\gamma(t))\, z'(t) \Big)\, dt$$

... and so on in higher dimensions.

Examples

* $\gamma(t) = (\underbrace{t}_{x(t)}, \underbrace{t^2}_{y(t)})$, $\quad 0 \le t \le 1.$



$\vec{F}(x,y) = (y, x)$, that is $P(x,y) = y$
$Q(x,y) = x$.

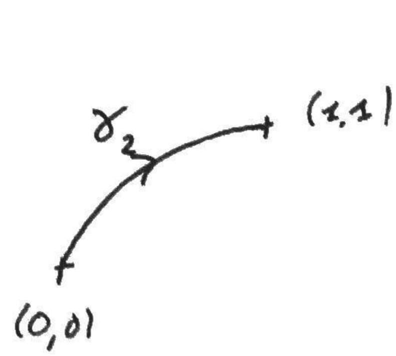
$$I = \int_\gamma \vec{F} \cdot ds = \int_0^1 \big(y(t)\, x'(t) + x(t)\, y'(t) \big)\, dt$$

$$= \int_0^1 \big(t^2 \cdot 1 + t \cdot 2t \big)\, dt = 3 \int_0^1 t^2\, dt = 1.$$

* γ_2 defined by

$$\begin{cases} x(t) = 1 - \cos t \\ y(t) = \sin t \end{cases} \quad t \in \left[0, \tfrac{\pi}{2}\right]$$

$\hookrightarrow \; (x-1)^2 + y^2 = 1$:



same $\vec{F}$

$$I = \int_{\gamma_2} \vec{F} \cdot ds = \int_0^{\frac{\pi}{2}} \big(y(t)\, x'(t) + x(t)\, y'(t) \big)\, dt$$

$$= \int_0^{\pi/2} \big(\sin^2(t) + (1 - \cos t) \cos t \big)\, dt$$

$$= \int_0^{\pi/2} \cos t \, dt - \int_0^{\pi/2} \cos(2t)\, dt$$

$$= \left[\sin t \right]_0^{\pi/2} - \left[\frac{1}{2} \sin(2t) \right]_0^{\pi/2} = 1 - 0 = 1 .$$

<u>Remark</u>: sometimes, we also write

$$\int_\gamma \vec{F} \cdot ds = \int_a^b P\, dx + Q\, dy .$$

<u>Properties</u>

· The definition of $\int_\gamma \vec{F} \cdot ds$ is independent of the parametrization of γ:

<u>Proof</u>: let $\tilde{\gamma}(\tilde{t}) = (\tilde{x}(\tilde{t}), \tilde{y}(\tilde{t}))$, $c \leq \tilde{t} \leq d$

be another parametrization (also regular).

Let $\varphi : [a, b] \to [c, d]$ be the strictly increasing and continuously diff. map such that

$$\forall t \in [a, b], \quad \gamma(t) = \tilde{\gamma}(\varphi(t)) .$$

$$\text{“} \varphi(t) = \tilde{t} \text{”}$$

change of variable

we have $\begin{cases} x(t) = \tilde{x}(\varphi(t)) \\ y(t) = \tilde{y}(\varphi(t)) \end{cases} \Rightarrow \begin{cases} x'(t) = \tilde{x}'(\varphi(t))\, \varphi'(t) \\ y'(t) = \tilde{y}'(\varphi(t))\, \varphi'(t) \end{cases}$

$$\int_c^d \left(P(\tilde{x}(\tilde{t}), \tilde{y}(\tilde{t}))\, \tilde{x}'(\tilde{t}) + Q(\tilde{x}(\tilde{t}), \tilde{y}(\tilde{t}))\, \tilde{y}'(\tilde{t}) \right) d\tilde{t}$$

$$\underset{\tilde{t} = \varphi(t)}{=} \int_a^b \left(P(\tilde{x}(\varphi(t)), \tilde{y}(\varphi(t)))\, \tilde{x}'(\varphi(t)) + Q(\tilde{x}(\varphi(t)), \tilde{y}(\varphi(t)))\, \tilde{y}'(\varphi(t)) \right) \varphi'(t)\, dt$$

$$= \int_a^b \left(P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t) \right) dt .$$

$$= \int_\gamma \vec{F} \cdot ds .$$

$\square$

- Let γ^R be the time-reversal of γ.

Then $\boxed{\int_{\gamma^R} \vec{F} \cdot ds = -\int_\gamma \vec{F} \cdot ds}$.

Proof: Let $r(s) = (x(s), y(s))$ be the natural par. of γ.
$s \leq 0 \leq L$.

Then $\tilde{r}(s) = (x(L-s), y(L-s))$ is ———— γ^R.

$$\int_{\gamma^R} \vec{F} \cdot ds = \int_0^L \vec{F}(\tilde{r}(s)) \cdot \tilde{r}'(s)\, ds$$

$$= \int_0^L \left(P(x(L-s), y(L-s)) (-x'(L-s)) + Q(x(L-s), y(L-s)) (-y'(L-s)) \right) ds$$

$$\overset{t = L-s}{=} -\int_0^L \left(P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t) \right) dt .$$

$$= -\int_\gamma \vec{F} \cdot ds .$$

$\square$