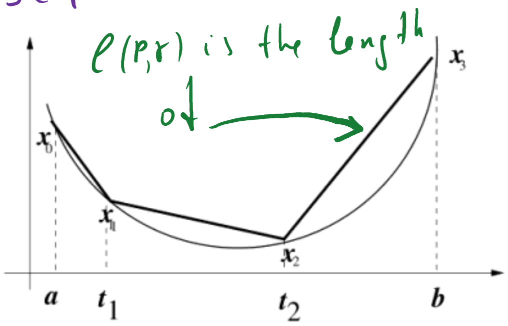
N6 Line integral of scalar field

1. Rectifiable Curves

The goal of this section is to define the length of a curve. So, let

$$\gamma : [a,b] \to \mathbb{R}^d$$

be a curve. We consider a partition $P = \{t_0, \dots, t_n\}$, where $a = t_0 < \dots < t_n = b$ and set



$$\ell(P,\gamma) := \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|$$

Def 6.1 A curve γ is said to be *rectifiable* if the supremum

$$\ell(\gamma) := \sup_P \ell(P,\gamma)$$

is finite, where the supremum is taken over all partitions P of $[a,b]$.

Prop 6.1 If γ' is continuous on $[a,b]$, then γ is rectifiable, and

$$\ell(\gamma) = \int_a^b \|\gamma'(t)\|\, dt.$$

Proof.

Let $\gamma(t) = (x_1(t), \ldots, x_d(t))$.

Then

$$\ell(P,\gamma) = \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\| =$$

$$= \sum_{i=1}^{n} \sqrt{(x_1(t_i) - x_1(t_{i-1}))^2 + \ldots + (x_d(t_i) - x_d(t_{i-1}))^2}$$

$$\approx \sum_{i=1}^{n} \sqrt{(x_1'(\xi_i))^2 + \ldots + (x_d'(\xi_i))^2}\,(t_i - t_{i-1}),$$

where $\xi_i \in [t_{i-1}, t_i]$, by the Lagrange theorem (See Th 11.4, Math 1)

Hence, if $\lambda(P) \to 0$, we have

$$\ell(P,\gamma) \to \int_a^b \sqrt{(x_1'(t))^2 + \ldots + (x_d'(t))^2}\, dt =$$

$$= \int_a^b \|\gamma'(t)\|\, dt.$$

$\blacksquare$

Rem. 6.1 If $d=2$, $\gamma(t) = (x(t), y(t))$, $t \in [a,b]$. Then

$$\ell(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

In particular, if γ is a graph of f, i.e.

$$\gamma(t) = (t, f(t)),\quad t \in [a,b],$$

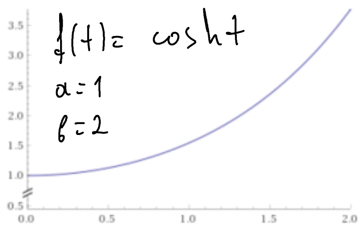
then

$$\ell(\gamma) = \int_a^b \sqrt{1 + (f'(t))^2}\, dt.$$

Example 6.1 a) Let γ be the graph of the function

$$f(t) = a\cosh\frac{t}{a}, \quad t\in[0,b],\ b>0,\ a\neq 0,$$

that is $\gamma(t) = (t, f(t)),\ t\in[0,b]$.



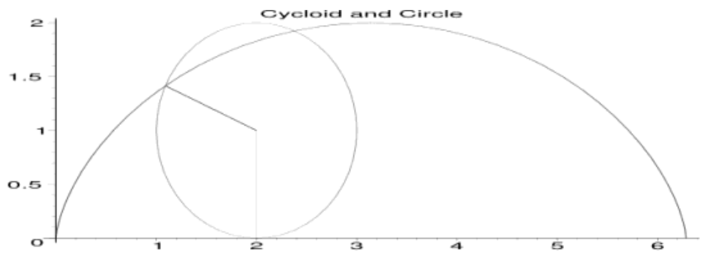
We compute

$$\ell(\gamma) = \int_0^b \sqrt{1+\left(\sinh\frac{t}{a}\right)^2}\,dt = \int_0^b \cosh\frac{t}{a}\,dt = a\sinh\frac{t}{a}\Big|_0^b$$

$$= a\sinh\frac{b}{a}.$$

b) (Length of cycloid)

Let $\gamma(\theta) = \big(a(\theta-\sin\theta),\ a(1-\cos\theta)\big),\ \theta\in[0,2\pi]$



γ is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a strain line.

So, we can compute

$$\ell(\gamma) = \int_0^{2\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\,d\theta = a\int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta$$

$$= a\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta = 2a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta =$$

$$= -2a\cdot 2\cos\frac{\theta}{2}\Big|_0^{2\pi} = 4a(-\cos\pi+\cos 0) = 8a.$$

Example 6.2 (A non-rectifiable curve) We consider

$\gamma(t) = (t, f(t)),\ t\in[0,1]$, the graph of

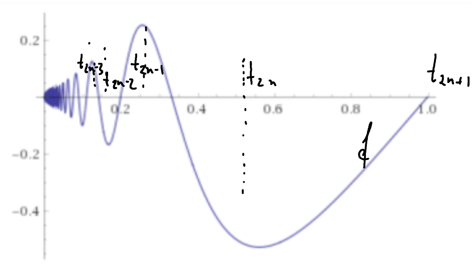
$$f(t) = \begin{cases} t\cos\frac{\pi}{2t}, & 0<t\leq 1 \\ 0, & t=0 \end{cases}$$

Since $\lim\limits_{t\to 0} f(t) = 0 = f(0)$, f is continuous and consequently γ is a curve. However, this curve is not rectifiable. Indeed, take the partition

$$P_n = \left\{ 0, \frac{1}{4n}, \frac{1}{4n-2}, \dots, \frac{1}{4}, \frac{1}{2}, 1 \right\},$$

here

$$t_i = \frac{1}{4n-2i+2}, \quad i = 1, \dots, n.$$



Then, $\left(\cos\frac{\pi}{2t_i}\right) = (1, -1, 1, -1, \dots, 1, -1)$

So, $\ell(P_n, \gamma) \geq \sum\limits_{i=2}^{2n} \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2}$

$$\geq \sum_{i=2}^{2n} |f(t_i) - f(t_{i-1})| \geq$$

$$\geq \left(\frac{1}{4n-2} + \frac{1}{4n}\right) + \left(\frac{1}{4n-4} + \frac{1}{4n-2}\right) + \dots + \left(\frac{1}{2} + \frac{1}{4}\right) =$$

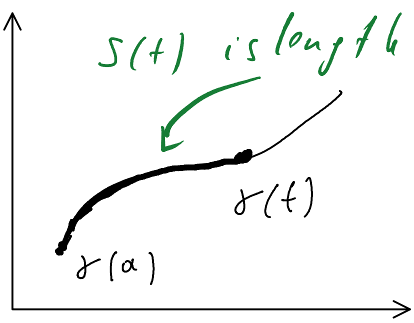
$$= \frac{1}{2} + 2\left(\frac{1}{4} + \dots + \frac{1}{4n-2}\right) + \frac{1}{4n} \to \infty, \quad n \to \infty.$$

2. Natural parametrization of rectifiable curve

Let $\gamma : [a, b] \to \mathbb{R}^d$ be a regular curve, that is, $\gamma'(t) \neq 0 \ \forall t \in [a, b]$. We denote by

$$s(t) = \int_a^t \|\gamma'(\tau)\| \, d\tau =: \ell_t(\gamma)$$

the length of a part of the curve $\gamma(\tau)$, $\tau \in [0, t]$



Since $\gamma'(t)\neq 0$, $\|\gamma'(t)\|>0$ and hence the function $s=s(t)$, $t\in[a,b]$ strictly increases. Consequently, s is invertible with inverse

$$t=t(s),\quad s\in[0,\ell],$$

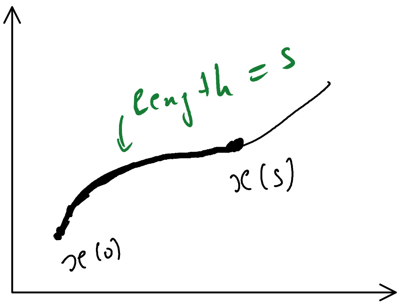
and $\ell=\ell(\gamma)=s(b)$.

We define

$$x(s)=\gamma(t(s)),\quad s\in[0,\ell]. \qquad (6.1)$$

It is another parametrisation of the curve γ.

Lemma 6.1. The length $\ell_s(x)$ of the curve given by $x=x(\tau)$, $\tau\in[0,s]$, equals s



Proof. We remark that

$$\gamma(t)=x(s(t)),\quad t\in[a,b].$$

By the chain rule

$$\|\gamma'(t)\|=\|x'(s(t))\|\cdot\|s'(t)\|=$$

$=\|x'(s(t))\|\cdot\|\gamma'(t)\|$. Since $\|\gamma'(t)\|\neq 0$, we have

$$\|x'(s(t))\|=1,\quad\forall t\in[a,b],$$

and hence

$$\|x'(s)\|=1,\quad\forall s\in[0,\ell].$$

Hence $\ell_s(x)=\int_0^s\|x'(\tau)\|\,d\tau=\int_0^s 1\,d\tau=s$. ∎

Def 6.2. The parametrization x defined by (6.1) is called the natural parametrization.

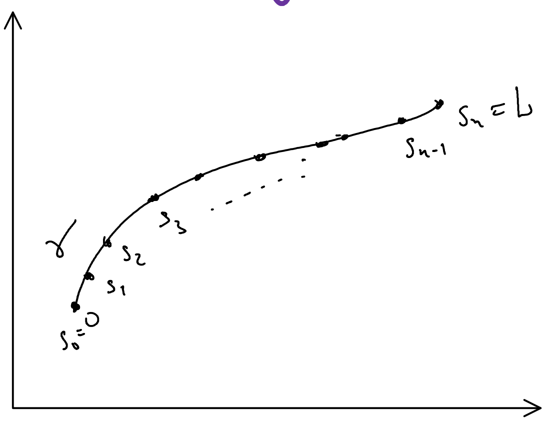
We remark that any regular curve has a natural parametrization.

3 Line integral of a scalar field

Let γ be a rectifiable curve with the length L. We assume that γ has a natural parametrization $x(s)$, $s \in [0, L]$. We set

$$\Gamma = \{x(s),\ s \in [0, L]\} = \{\gamma(t),\ t \in [a, b]\}$$

Let $f : \Gamma \to \mathbb{R}^d$. We are going to introduce the integral f over γ



We take a partition $P = \{s_0, s_1, \dots, s_n\}$ of $[0, L]$ we define the line integral as

$$\int_\gamma f\, ds := \lim_{\lambda(P) \to 0} \sum_{i=1}^{n} f(x(s_i))(s_i - s_{i-1})$$

if the limit exists.

Rem 6.2 The line integral $\int_\gamma f\, ds$ coinside with the usual Riemann integral

$$\int_0^L f(x(s))\, ds,$$

where x is a natural parametrization of γ.