N5. Improper integral

1. Improper integral.

In this section, we are going to define the Riemann integral over unbounded sets. We start from a definition

Def 5.1 An *exhaustion* of a set $S \subseteq \mathbb{R}^d$ is a sequence of (Jordan) measurable sets $\{S_n\}$ such that $S_n \subseteq S_{n+1} \subseteq S$ for any $n \geq 1$ and $\bigcup_{n=1}^{\infty} S_n = S$.

Lemma 5.1 If $\{S_n\}$ is an exhaustion of a measurable set S, then:

a) $\lim_{n\to\infty} \mu(S_n) = \mu(S)$ (here $\mu(S) = \int_S dx = \int_I \mathbb{1}_S \, dx$)

b) for every $f \in \mathcal{R}(S)$, f also belongs to $\mathcal{R}(S_n)$ and

$$\lim_{n\to\infty} \int_{S_n} f(x)\, dx = \int_S f(x)\, dx.$$

Proof We will give only the idea of the proof. We start from a).

Since the sequence S_n increases, $\mu(S_n)$ also increases by Corollary 3.1. Indeed,

$\mathbb{1}_{S_n} \leq \mathbb{1}_{S_{n+1}}$, and let $I \supseteq S_{n+1}$ be an interval. So,

$$\mu(S_n) = \int_{S_n} dx = \int_I \mathbb{1}_{S_n}(x)\, dx \overset{\text{Cor. 3.1}}{\leq} \int_I \mathbb{1}_{S_{n+1}}(x)\, dx = \mu(S_{n+1})$$

Moreover, the sequence $\{\mu(S_n)\}_{n\geq 1}$ is bounded above by $\mu(S)$. Hence, there exists a limit

$$\lim_{n\to\infty} \mu(S_n) \leq \mu(S).$$

The proof of the inequality

$$\lim_{n\to\infty} \mu(S_n) \geq \mu(S) \qquad (5.1)$$

is more complicated and is based on properties of compact sets. We omit it here. (The complete proof of (5.1) can be found in [Zorich MA II, p.150])

Now we prove b). We first remark that the fact that $f \in \mathcal{R}(S_n)$ follows from Prop 3.2 a). Since $f \in \mathcal{R}(S)$, f is bounded on S, by Prop 1.1. So, let $|f| \leq C$, where C is a constant. Then

$$\left| \int_S f(x)dx - \int_{S_n} f(x)\,dx \right| = \left| \int_{S\setminus S_n} f(x)\,dx \right| \leq$$

$$\leq \int_{S\setminus S_n} |f(x)|dx \leq \int_{S\setminus S_n} M\,dx = M\,\mu(S\setminus S_n)$$

Since, $\mu(S\setminus S_n) = \mu(S) - \mu(S_n) \to 0$, we obtain the statement b).

Def 5.2 Let $\{S_n\}$ be an exhaustion of the set S and suppose that $f: S \to \mathbb{R}$ be integrable on S_n for all $n \geq 1$. If the limit

$$\int_S f(x)\,dx := \lim_{n\to\infty} \int_{S_n} f(x)\,dx \qquad (5.2)$$

exists and does not depend on the choice of the sets in the exhaustion of S. This limit is called the improper integral of f over S.

We will also say that the integral exists or converges if the limit in (5.2) exists.

Rem. 5.1 If S is a (Jordan) measurable and $f \in \mathcal{R}(S)$, then the integral of f over S in the sence of Def. 5.2 exists and has the same value as the proper integral of f over E.

The remark follows from Lemma 5.1 b).

Prop 5.1 If $f: S \to \mathbb{R}$ is nonnegative and the limit (5.2) exists for even one exhaustion $\{S_n\}$ of S, then the improper integral of f over S converges. (For the proof see [Zorich MA II, p 152])

Examples 5.1 Let us compute

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy.$$

We will exhaust the plane $\mathbb{R}^2$ by the sequence

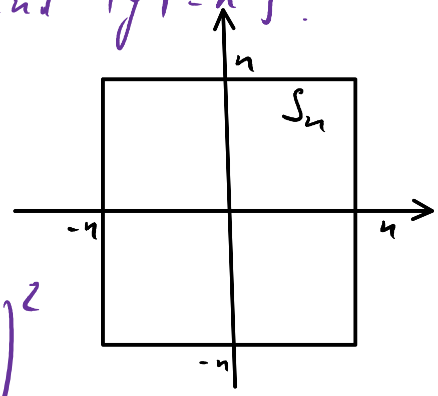
$S_n := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < n^2\}$.

After passing to the polar coordinates, we find

$$\iint_{S_n} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^n e^{-r^2} r\,dr = \pi(1-e^{-n^2}) \to \pi, \quad n\to\infty.$$

Hence, $$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \pi.$$

We remark, that considering other exhausion of the plane: $S_n' = \{(x,y) : |x| \le n \text{ and } |y| \le n\}$.



Then

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \lim_{n\to\infty} \iint_{S_n'} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \lim_{n\to\infty} \int_{-n}^{n} dy \int_{-n}^{n} e^{-(x^2+y^2)}\,dx = \lim_{n\to\infty} \left(\int_{-n}^{n} e^{-x^2}\,dx\right)^2$$

$$= \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2.$$ Hence, we can conclude that

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$ (This integral is called the Euler-Poisson integral)

Rem 5.2 1) Improper integral can arise if S is unbounded or if f is unbounded

2) Various properties of multiple integral can

can be suitably extended to improper integrals.

Th 5.1 (the comparison test). Let f and g be functions defined on S and integrable over exactly the same measurable subsets of it, and suppose that $|f(x)| \leq g(x)$, $\forall x \in S$.

If the improper integral $\int_S g(x)dx$ converges, then integrals $\int_S |f(x)|dx$ and $\int_S f(x)dx$ also converge.

For proof see [Zorich MA II, p 153].

2. Curves in $\mathbb{R}^d$.

Our further goal the integration over curves. So, in this section, we will give the definition of a curve, tangent vector, regular points etc.

Def 5.3 A curve in $\mathbb{R}^d$ is a continuous map

$$\gamma : I \to \mathbb{R}^d,$$

where I is a closed interval consisting more than one point. (I could be $[a,b]$, $a<b$; $[a,+\infty)$, $(-\infty, b]$, $\mathbb{R}$)

If $I=[a,b]$, then a is called the initial point of γ

and b is called the end point of γ.
These two points define a natural orientation of γ from $\gamma(a)$ to $\gamma(b)$. Replacing $\gamma(t)$ with $\gamma(a+b-t)$ we obtain the curve with opposite orientation.

If $\gamma(a)=\gamma(b)$, γ is said to be a closed curve. If γ is differentiable, the curve is said to be differentiable.

If γ has no self-intersection, i.e. γ is injective on I°, then γ is said to be simple.

Example 5.2 a) A circle in $\mathbb{R}^2$ of radius $r>0$ with center $(0,0)$ is discribed by the curve

$$\gamma(t)=(r\cos t,\ r\sin t),\quad t\in[0,2\pi].$$

Note that

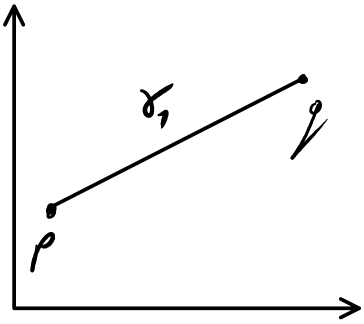
$$\tilde{\gamma}(t)=(r\cos t,\ r\sin t),\quad t\in[0,4\pi]$$

has the same image, but is different from γ.
γ is simple, $\tilde{\gamma}$ is not.

b) Take $p,q\in\mathbb{R}^d$, $p\neq q$. Define two curves:

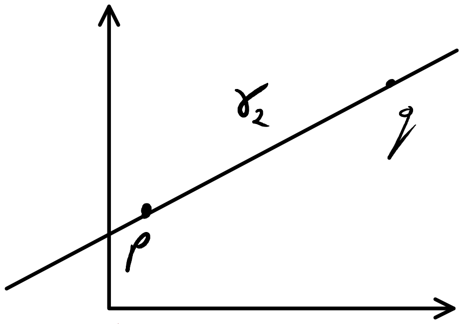
$$\gamma_1(t)=(1-t)p+tq,\quad t\in[0,1],$$
$$\gamma_2(t)=(1-t)p+tq,\quad t\in\mathbb{R}.$$

γ_1 is the segment from p to q.
γ_2 is the line through p and q.



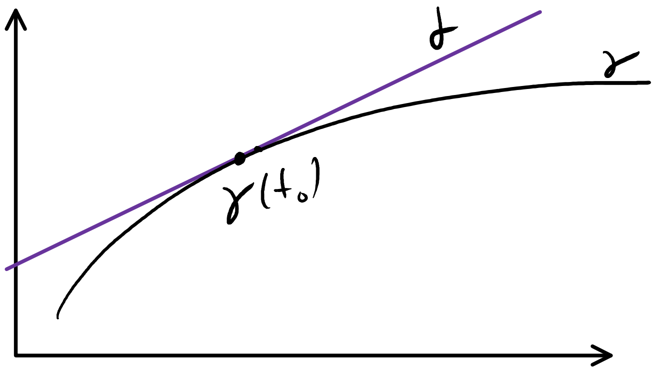
c) If $f:[a,b] \to \mathbb{R}$ is a continuous function, the graph of f is a curve in $\mathbb{R}^2$:

$$\gamma(t) = (t, f(t)), \quad t \in [a,b].$$



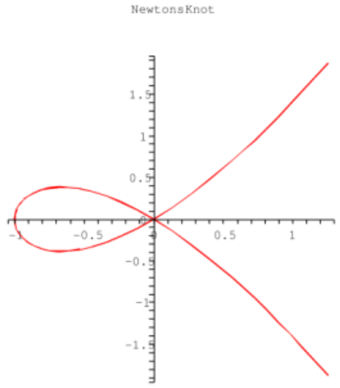
Def 5.4 a) A simple curve $\gamma: I \to \mathbb{R}^d$ is called regular at t_0 if γ is continuously differentiable on I and $\gamma'(t_0) \neq 0$. γ is regular if γ is regular at any point $t_0 \in I$.

b) The vector $\gamma'(t_0)$ is called the tangent vector, and $\alpha(t) = \gamma(t_0) + t\,\gamma'(t_0)$ is called the tangent line to γ at $\gamma(t_0)$



Example 5.3 (Newton's knot) We consider the curve

$$\gamma(t) = (t^2 - 1, t^3 - t), \quad t \in \mathbb{R}$$



The curve γ is not simple, since it is not injective $\gamma(-1)=\gamma(1)=(0,0)$. The point $(0,0)$ is a double point and γ has two different tangent lines at this point.