Lecture N3. Fubini's theorem

1. General properties of the integral.

Let $\mathcal{R}(S)$ denote the set of integrable functions $f: S \to \mathbb{R}$ over $S \subseteq \mathbb{R}^d$.

Prop. 3.1 if $f, g \in \mathcal{R}(S)$ and $a \in \mathbb{R}$, then $f+g$, af, $f \cdot g \in \mathcal{R}(S)$ and

$$\int_S (f+g)\,dx = \int_S f\,dx + \int_S g\,dx$$

$$\int_S af\,dx = a \int_S f\,dx.$$

Proof follows from the definition of the integral.

Prop. 3.2 Let S_1, S_2 be admissible sets in $\mathbb{R}^d$ and $f: S_1 \cup S_2 \to \mathbb{R}$

a) $f \in \mathcal{R}(S_1 \cup S_2) \iff f \in \mathcal{R}(S_1) \cap \mathcal{R}(S_2)$

b) if additionally $\mu(S_1 \cap S_2) = \int_{S_1 \cap S_2} dx = 0$, then

$$\int_{S_1 \cup S_2} f\,dx = \int_{S_1} f\,dx + \int_{S_2} f\,dx.$$

<u>Proof</u> a) follows from Lebesgue's criterion.
In order to prove b) we observe that

$$\mathbb{I}_{S_1 \cup S_2} = \mathbb{I}_{S_1} + \mathbb{I}_{S_2} - \mathbb{I}_{S_1 \cap S_2}$$

Hence, by Prop 3.1. for $I \supseteq S_1 \cup S_2$

$$\int_{S_1 \cup S_2} f \, dx = \int_I f \mathbb{I}_{S_1 \cup S_2} \, dx =$$

$$= \int_I f \mathbb{I}_{S_1} \, dx + \int_I f \mathbb{I}_{S_2} \, dx - \int_I f \mathbb{I}_{S_1 \cap S_2} \, dx =$$

$$= \int_I f \mathbb{I}_{S_1} \, dx + \int_I f \mathbb{I}_{S_2} \, dx = \int_{S_1} f \, dx + \int_{S_2} f \, dx.$$

The fact that $\int_I f \mathbb{I}_{S_1 \cap S_2} \, dx = 0$ follows from the following exercise. ∎

<u>Exercise 3.1</u> Let $f \in \mathcal{R}(S)$ and $f = 0$ almost everywhere on S. Show that $\int_S f \, dx = 0$.

Prop. 3.3 If $f \in \mathcal{R}(S)$, then $|f| \in \mathcal{R}(S)$ and

$$\left| \int_S f(x)\, dx \right| \le \int_S |f(x)|\, dx. \tag{3.1}$$

Proof. The relation $|f| \in \mathcal{R}(S)$ follows from the Lebesgue criterion since

$$D_{|f|} = \{ x : |f| \text{ is discontinuous at } x \} \subseteq$$
$$\subseteq D_f = \{ x : f \text{ is discontinuous at } x \}.$$

The inequality (3.1) follows from the corresponding inequality for Riemann sums. ∎

Prop 3.4 If $f \in \mathcal{R}(S)$ and $\forall x \in S$ $f(x) \ge 0$, then $\int_S f(x)\, dx \ge 0$.

The proposition follows from the equality

$$\int_S f(x)\, dx = \int_I \underbrace{f(x)\, \mathbb{1}_S(x)}_{\ge 0}\, dx$$

and the corresponding inequality for the Riemann sums.

Corollary 3.1 If $f, g \in \mathcal{R}(S)$, $f \le g$ on S, then $\int_S f(x)\,dx \le \int_S g(x)\,dx$

Corollary 3.2 If $f \in \mathcal{R}(S)$, $m \le f(x) \le M$, $\forall x \in S$, then

$$m\mu(S) \le \int_S f(x)\,dx \le M\mu(S).$$

2. Fubini's Theorem

Let $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$. We denote

$$X \times Y = \{(x, y) \in \mathbb{R}^{m+n} : x \in X,\ y \in Y\}$$

Th 3.1 (Fubini's theorem) Let $X \times Y$ be an interval in $\mathbb{R}^{m+n}$, where X, Y be intervals in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. If

$$f: X \times Y \to \mathbb{R}$$

is integrable over $X \times Y$, then

$$\iint_{X \times Y} f(x, y)\,dx\,dy = \int_X dx \int_Y f(x, y)\,dy =$$

$$= \int_Y dy \int_X f(x, y)\,dx.$$

Here $\int_X dx \int_Y f(x,y)\,dy$ and $\int_Y dy \int_X f(x,y)\,dx$ denotes the iterated integrals i.e.

$$\int_X dx \int_Y f(x,y)\,dy = \int_X \left[\int_Y f(x,y)\,dy \right] dx$$

$$\int_Y dy \int_X f(x,y)\,dx = \int_Y \left[\int_X f(x,y)\,dx \right] dy$$

Proof. For the proof of the theorem see [Zorich, MA II] p. 128 – 129.

Corollary 3.3 If $I = [a_1, b_1] \times \ldots \times [a_d, b_d] = I_{a,b}$, then

$$\int_I f(x)\,dx = \int_{a_d}^{b_d} dx_d \int_{a_{d-1}}^{b_{d-1}} dx_{d-1} \ldots \int_{a_1}^{b_1} f(x_1, \ldots, x_d)\,dx_1$$

Ex. 3.1 Let $f(x,y,z) = z \sin(x+y)$ and $I \subseteq \mathbb{R}^3$ defined by the relations

$$0 \le x \le \pi, \quad |y| \le \frac{\pi}{2}, \quad 0 \le z \le 1,$$

i.e. $I = \{ (x,y,z) : 0 \le x \le \pi, -\frac{\pi}{2} \le y \le \frac{\pi}{2}, 0 \le z \le 1 \}$

By Corollary 2

$$\iiint\limits_I f(x,y,z)\,dx\,dy\,dz = \int_0^1 dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy \int_0^{\pi} z\sin(x+y)\,dx =$$

$$= \int_0^1 dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-z\cos(x+y)\Big|_{x=0}^{x=\pi} \right) dy =$$

$$= \int_0^1 dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2z\cos y\,dy = \int_0^1 \left(2z\sin y\Big|_{y=-\frac{\pi}{2}}^{y=\frac{\pi}{2}} \right) dz =$$

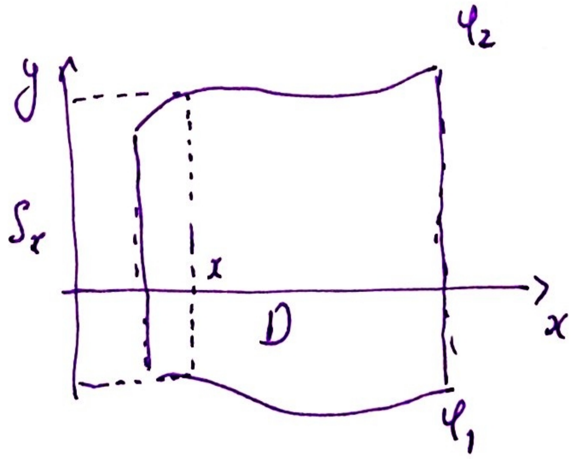
$$= \int_0^1 4z\,dz = 2.$$

<u>Corollary 3.4</u> Let D be a bounded set in $\mathbb{R}^{d-1}$ and $S = \{(x,y) \in \mathbb{R}^d : x \in D,\ \varphi_1(x) \le y \le \varphi_2(x)\}$. If $f \in \mathcal{R}(S)$, then

$$\iint\limits_S f(x,y)\,dx\,dy = \int_D dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy.$$

<u>Proof</u>

We set



$$S_x = \{y \in \mathbb{R} : \varphi_1(x) \le y \le \varphi_2(x)\}$$

if $x \in D$ and

$S_x = \emptyset$ if $x \in D$.

We remark, that

$$\mathbb{I}_S(x,y) = \mathbb{I}_D(x)\,\mathbb{I}_{S_x}(y).$$

⑦ We take an interval $I \supseteq S$ and use Th. 3.1.

$$\iint_S f(x,y)\,dx\,dy = \int_I f\,\mathbb{1}_S(x,y)\,dx\,dy =$$

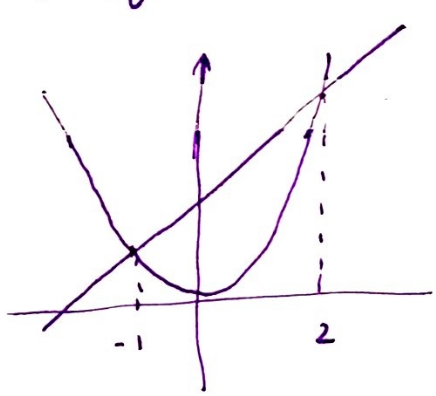
$$= \int_{I_x} dx \int_{I_y} f\,\mathbb{1}_S(x,y)\,dx\,dy = \int_{I_x}\left(\int_{I_y} f(x,y)\,\mathbb{1}_{S_x}(y)\,dy\right)\cdot$$

$$\cdot\,\mathbb{1}_D(x)\,dy$$

$$= \int_{I_x}\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right)\mathbb{1}_D(x)\,dx = \int_D dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy.$$

Ex. 3.2 S is bounded by the parabola $y = x^2$ and line $y = x+2$. $f(x,y) = x^2$.

$$S = \{(x,y) : -1 \le x \le 2,\ x^2 \le y \le x+2\}$$



$$\iint_S f(x,y)\,dx\,dy = \int_{-1}^{2} dx \int_{x^2}^{x+2} y^2\,dy =$$

$$= \int_{-1}^{2}\left(\frac{y^3}{3}\Big|_{y=x^2}^{y=x+2}\right) = \int_{-1}^{2}\left(\frac{(x+2)^3}{3} - \frac{x^6}{3}\right)dx =$$

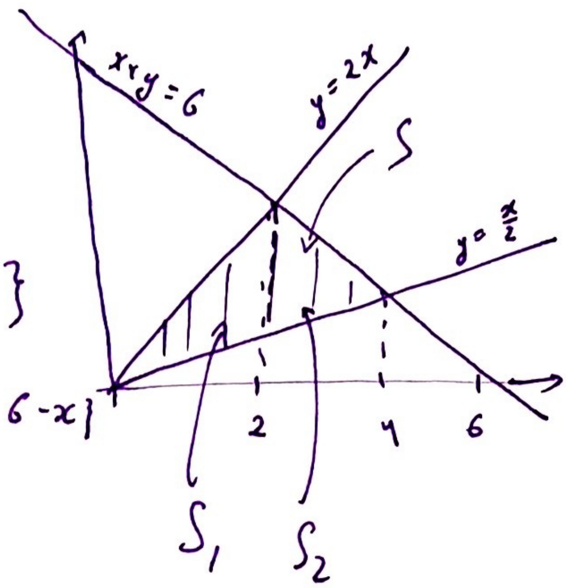
$$= \frac{1}{3}\left[\frac{(x+2)^4}{4}\Big|_{-1}^{2} - \frac{x^7}{7}\Big|_{-1}^{2}\right] = \frac{1}{3}\left[\frac{4^4}{4} - \frac{1}{4} - \frac{2^7}{7} - \frac{1}{7}\right] =$$

$$= \frac{423}{28}$$

⑧

Ex 3.3 S is bounded by $y = 2x$, $y = \frac{x}{2}$, $x+y=6$

$$f(x,y) = \frac{1}{(1+x+y)^2}$$



Take

$S_1 = \{(x,y) : 0 \le x \le 2,\ \frac{x}{2} \le y \le 2x\}$

$S_2 = \{(x,y) : 2 \le x \le 4,\ \frac{x}{2} \le y \le 6-x\}$

$$\iint_S f(x,y)\,dx\,dy = \int_{S_1} + \int_{S_2} =$$

$$= \int_0^2 dx \int_{\frac{x}{2}}^{2x} \frac{dy}{(1+x+y)^2} + \int_2^4 dx \int_{\frac{x}{2}}^{6-x} \frac{dy}{(1+x+y)^2} = \dots = \frac{1}{3}\ln 7 - \frac{2}{7}.$$

Ex 3.4 $f(x,y,z) = y$, $S = \{(x,y,z) : |x| \le z,\ 0 \le z \le 1,$
$z \le y,\ x^2+y^2+z^2 \le 4\}$

$$\iiint_S f(x,y,z)\,dx\,dy\,dz = \int_0^1 dz \int_{-z}^{z} dx \int_z^{\sqrt{4-x^2-z^2}} y\,dy = \dots = \frac{17}{12},$$

where

$S = \{(x,y,z) : |x| \le z,\ 0 \le z \le 1,\ z \le y \le \sqrt{4-x^2-y^2}\}$