Lecture N2 The integral over a set

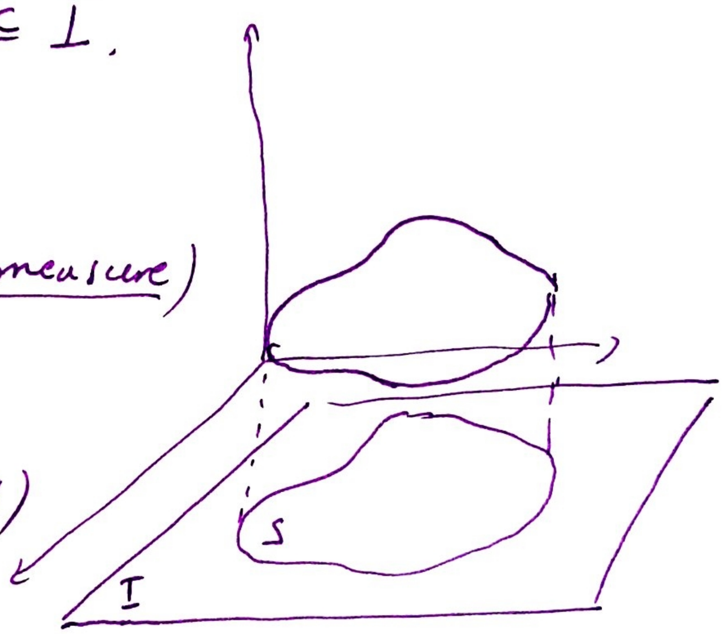
1. The measure (volume) of a set.

In this section we define the measure of a set. Let $S \subseteq \mathbb{R}^d$ be a bounded set. Let I be an interval in $\mathbb{R}^d$ such that $S \subseteq I$.

We can define

The <u>measure (Jordan measure)</u> of S is

$$\mu(S) = \int_I \mathbb{I}_S(x)\,dx \qquad (1)$$



if the integral exists,

where $\mathbb{I}_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \in S. \end{cases}$

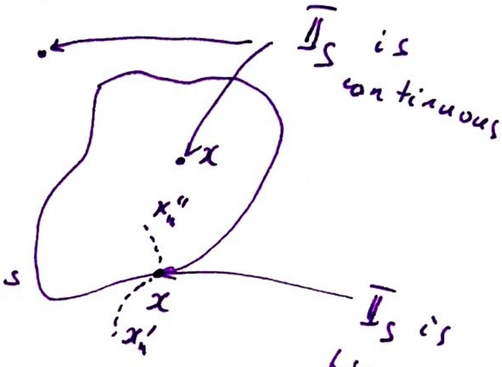
Let us figure out when the integral (1) exists.

By Lebesgue criterion (Th 1.2), integral (1) exists if $\mathbb{I}_S$ is continuous almost everywhere.

(2)

Let us investigate for which sets S $\mathbb{I}_S$ is continuous almost everywhere.

Let $D_{\mathbb{I}_S} = \{ x : \mathbb{I}_S \text{ is discontinuous at } x \}$



sinse $\exists\ x_n' \notin S$
$x_n' \to x$
$\mathbb{I}_S(x_n') = 0$

$\exists\ x_n'' \in S$
$x_n'' \to x$
$\mathbb{I}_S(x_n'') = 1.$

__Def. 2.1.__ The set

$$\partial S = \{ x : \forall \varepsilon > 0 \ B_\varepsilon(x) \cap S \neq \emptyset, \ B_\varepsilon(x) \cap S^c \neq \emptyset \},$$

where $B_\varepsilon(x) = \{ y \in \mathbb{R}^d : \| x - y \| < \varepsilon \}$ denotes the ball with center x and radius ε, $S^c = \mathbb{R}^d \setminus S$, is called __the boundary of S__

__Ex. 2.1__ Prove that $\partial S = \bar{A} \setminus A^\circ$, where $\bar{A}$ is the closure of A and A° is the interior of A.

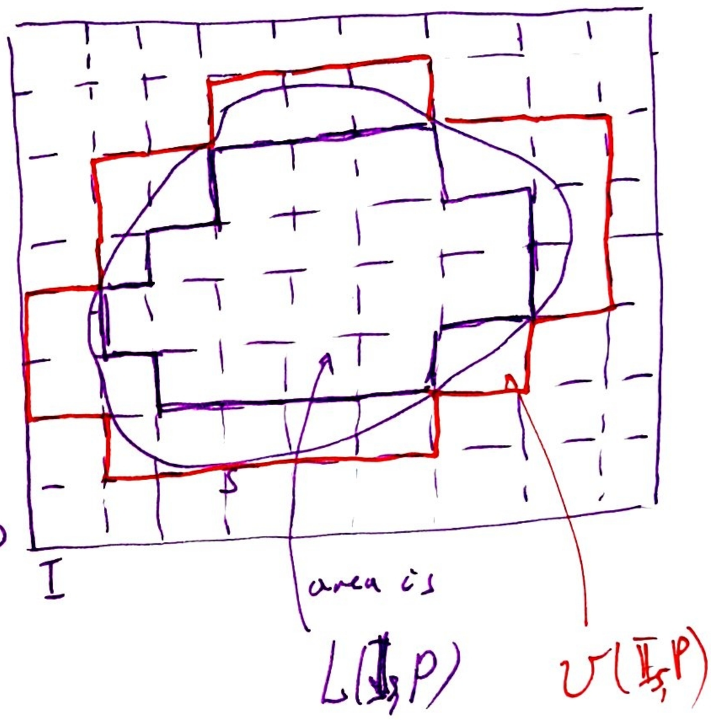
__Lemma 2.1__ The set $D_{\mathbb{I}_S}$ coinsides with ∂S.

$$\boxed{\text{Hence, by Lemma 2.1 the measure } \mu(S) \text{ of a set } S \subseteq \mathbb{R}^d \text{ exists iff the boundary } \partial S \text{ of } S \text{ has measure zero}}$$

③

Let us explain the meaning of $\mu(E)$.
Let us consider $S \subseteq \mathbb{R}^d$ and ∂S has measure zero.

We recall, that



$$\mu(S) = \int_I \mathbb{I}_S(x)\, ds =$$

$$= \overline{J} = \underline{J}, \text{ where}$$

$\overline{J}$ and $\underline{J}$ are upper and lower Darboux integrals, i.e.

$$\underline{J} = \sup_P L(\mathbb{I}_S, P)$$

$$\overline{J} = \inf_P U(\mathbb{I}_S, P)$$

and $L(\mathbb{I}_S, P) = \sum_i m_i |I_i|, \quad m_i = \inf_{x \in I_i} \mathbb{I}_S(x)$

$$U(\mathbb{I}_S, P) = \sum_i M_i |I_i|, \quad M_i = \sup_{x \in I_i} \mathbb{I}_S(x)$$

We remark, that $L(\mathbb{I}, P)$ coinsides with volume of intervals of partition P which belongs to S, and $U(\mathbb{I}, P)$ coinsides with volume of intervals of partition P which cover S.

By Darboux cratirion (Th 1.1) this two volumes converges to the same number $\mu(S)$ which is called the measure (or volume) of a set S.

(This measure is also called the Jordan measure and the set S is Jordan-measurable set. Leter in Math 4 we consider wider notion od measurable sets called Lebesgue measurable sets)

2. The integral over a set

Def 2.2 A set $S \subseteq \mathbb{R}^d$ is admissible if it is bounded in $\mathbb{R}^d$ and ∂S has measure zero.

Def 2.3 The integral of f over S is given by

$$\int_S f(x)\, dx := \int_I f(x) \mathbb{I}_S(x)\, dx, \quad (2)$$

where I is some interval in $\mathbb{R}^d$, $S \subseteq I$.

if the integral (2) exists then f is called (Riemann) integrable over S.

Now we discuss which functions are integrable.

<u>Lemma 2.2</u> For any S, S_1, S_2

a) ∂S is closed in $\mathbb{R}^d$

b) $\partial(S_1 \cup S_2) \subset \partial S_1 \cup \partial S_2$

c) $\partial(S_1 \cap S_2) \subset \partial S_1 \cup \partial S_2$

d) $\partial(S_1 \setminus S_2) \subset \partial S_1 \cup \partial S_2$.

<u>Proof</u> a) it follows from the fact that

$\partial S = \bar{S} \setminus S^\circ$.

b) We first note that $x \in \partial S$ iff $\exists\ x_n' \in S$ s.t. $x_n' \to x$ and $\exists\ x_n'' \notin S$ s.t. $x_n'' \to x$.

Let $x \in \partial(S_1 \cup S_2)$ then $\exists\ x_n' \in S_1 \cup S_2$, $x_n' \to x$ and $\exists\ x_n'' \notin S_1 \cup S_2$, $x_n'' \to x$.

Trivially, there exists a subsepueness x_{n_k}' which belongs either to S_1 or to S_2. Let it belongs to S_1, then $x_{n_k}' \in S_1$, $x_{n_k}' \to x$.

So; $x \in \partial S_1 \Rightarrow x \in \partial S_1 \cup \partial S_2$. ∎

Ex 2.2. Prove c) and d).

Lemma 2.3 The union or intersection of a finite number of admissible sets is an admissible set. The difference of an admissible sets is also an admissible set.

Proof. Let $S_1, S_2, \dots, S_n$ be admissible, then they are bounden and ∂S_i, $i=1,\dots,n$ have measure zero. By Lemma 2.2

$$\partial\left(\bigcup_{i=1}^{n} S_i\right) \subseteq \bigcup_{i=1}^{n} \partial S_i .$$

Using Lemma 1.2 a) $\partial\left(\bigcup_{i=1}^{n} S_i\right)$ has measure zero. Moreover, $\bigcup_{i=1}^{n} S_i$ is bounded. So $\bigcup_{i=1}^{n} S_i$ is admissible. The similar argument works for intersection and difference. ∎

<u>Theorem 2.1</u> A function $f : S \to \mathbb{R}$ is integrable over an admissible set S if and only if it is bounded and continuous almost everywhere.

<u>Proof.</u> Let $I \supseteq S$ be an interval on $\mathbb{R}^d$. The function $f \cdot \mathbb{I}_S : I \to \mathbb{R}$ can have only additional points of discontinuity only on the boundary ∂S of S, which has measure zero. ∎